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Semi-endogenous growth models with domestic and foreign private and public R&D linked to VECMs with evidence for five countries

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Abstract We present semi-endogenous growth models with productivity as functions of domestic and foreign private and public R&D. In a small country case with a Cobb-Douglas productivity production function, foreign R&D drives steady-state growth and the production function can be a long-term relation in a vector-error-correction model (VECM). Marginal productivity conditions can be long-term relations for a vector-error-correction model if the functional form is of a VES function generalizing a CES function. Combining the marginal products of VES functions with recent evidence from VECMs for five countries shows that private and public R&D have a positive effect on productivity (except for France), and a negative R&D augmenting technical change. In case of a VES function, steady states with constant R&D/productivity ratios exist only for special cases of parameter restrictions, which are not supported by the evidence.

Keywords: Productivity, endogenous (un)balanced growth, public R&D expenditure, foreign spillover. JEL code: O38, O40, O41, H54, H87.

1. Introduction

Empirical literature has found that public R&D stimulates private R&D or there is incomplete crowding out (Leyden and Link 1991, Guellec and van Pottelsberghe de la Potterie 2003; Jaumotte and Pain 2005a, b; Ziesemer 2019; Szücs 2020). There is a positive impact of domestic and foreign stocks of aggregate gross R&D expenditure and domestic business R&D on TFP (Luintel and Khan 2004) as well as of R&D disaggregated into business and public R&D on TFP (Guellec and van Pottelsberghe de la Potterie 2004; Khan and Luintel 2006). These papers use single equation regressions to find statistically significant coefficients for the link from public to private R&D and from private R&D to TFP.[1] In recent work, Soete et al. (2020a,b) estimate the link between GDP, TFP, domestic and foreign private and public R&D using a vector-error-correction (VEC) approach. In this way, they find dynamic country-specific models, where the six variables interact, allowing for the analysis of permanent policy shocks on public R&D and avoiding many problems of single equation regressions. As this approach is a method of 'letting the data speak' (Juselius 2006), the theoretical basis remains unclear or intuitive and it seems desirable to have a theoretical underpinning for it, which is what we try to contribute in this paper linking theory and evidence from dynamic models.

[1] In the case of Luintel and Khan (2004) this is a cointegrating equation in a VECM.

The related theoretical literature has linked public and private knowledge in Shell (1967), Ziesemer (1990, 1995), Antonelli (2019). However, they do not estimate these theoretical models. Conversely, Coccia (2008, 2010) has an empirical model, but no theoretical foundation. McMorrow and Röger (2009) calibrate a DSGE model with an R&D part from Jones (1995). This does not include publicly performed R&D nor does it question the assumption of the Cobb-Douglas function for R&D with its unit elasticity of substitution, as we do in this paper. In short, in this literature either evidence, or theory or crucial variables are missing.

Some authors link theoretical and empirical models including public and private R&D and TFP. Park (1995) provides evidence for ten OECD countries suggesting that private R&D stocks improve domestic and foreign output via productivity. Moreover, global and foreign public as well as foreign (mainly US) private R&D stocks stimulate domestic privately financed R&D. Park (1998) extends the model of Romer (1990) with private domestic TFP depending on public and foreign R&D. All these variables depend on each other. The evidence is used to identify the elasticities of production of a Cobb-Douglas function for final output.[2] Results with Cobb-Douglas functions imply that that there is a unit elasticity of substitution for any two variables of labour, capital and R&D variables, although the literature on capital-labour substitution has lower elasticities (Ziesemer 2020, footnote 4; Knoblach et al. 2020; Jiang et al. 2019). Estrada and Montero (2009) set up a log-linear theoretical model with private and public R&D. They add residuals and autoregressive processes for the residuals and solve in the form of a structural VAR. The parameter estimates of the theoretical models are not retrieved and therefore we cannot see whether the employed Cobb-Douglas functions are valid.[3] Akcigit et al. (2016) estimate the parameters of a closed economy

[2] Public R&D variables have negative coefficients if other R&D variables are added. Domestic private R&D becomes insignificant when foreign private R&D is included unless investment is lagged more than three periods in the accumulation process. Foreign private R&D is insignificant for Japan, Germany, USA. When cumulated investment is lagged more than three years before entering the stock, public R&D is not considered; the elasticity of production is around 0.1 for private R&D and between 0.1 and 0.45 for foreign private R&D. Rates of return are higher for domestic than foreign private R&D. The authors provide strong indications for panel heterogeneity. Collinearity is not tested for. Whenever different types of R&D have an impact on each other, for example public on private R&D (Sveikauskas 2007), they should never both appear on the right-hand side of a regression, but rather econometric textbooks recommend using simultaneous equation models.

[3] They estimate the SVAR in first differences. This should be done only in the absence of cointegration, for which they do not test. This may lead to loss of information from lack of cointegrating equations, except perhaps for fitting of short-run adjustments where long-run relations may work like constraints even with non-zero residual values (Gospodinov et al. 2013). With estimation in differences they should not include time trends, because the long-term out-of-sample simulations get unrealistic with even only small trends in growth rates and the estimates may get blurred. Foreign R&D is not split up into private and public, unlike Soete et al. (2020a,b), and has no effect.

endogenous growth model with basic and applied research using data for France; the assumption of Cobb-Douglas functions requires a unit elasticity of substitution for intangible capital and research labour, which is imposed, not estimated. Cantore (2018) tries to link the evidence from a structural VAR including publicly financed R&D to a theoretical model. This intention comes close to our idea of linking the empirical to the theoretical model. However, he does not use the SVAR evidence for the calibration of the theoretical model;[4] private R&D or TFP are not included.

In order to link theory and evidence regarding the relation between TFP, public and private R&D, without insisting on Cobb-Douglas productivity functions, we develop (semi-) endogenous growth models that may have long-run relationships that we can relate to long-term cointegrating equations of VEC models. We explore the long-term relations for two alternative productivity production functions with domestic and foreign public and private R&D as inputs. The Cobb-Douglas function is a special case of the CES (constant elasticity of substitution) function. Mukerji's (1963) VES (variable elasticity of substitution) function generalizes the CES function by way of replacing the CES parameter by several different parameters. We add R&D augmenting technical change and find the VES parameters by comparison of the VES function's marginal products with the VECM regression coefficients.

As a by-product of our intentions, we also contribute to the literature on technical change in the production function for knowledge. The surveys by Bayar et al. (2007) and Neves and Sequeira (2017) show that the issue is hardly recognized although the authors are aware of the conceptual possibility. This topic is important, also because it is not discussed in endogenous growth theory or the TFP regressions mentioned above.[5] Some papers discuss the issue though. Park (1995) replaces total factor productivity in the output production function by

[4] The major result is that a shock to public R&D strongly decreases the labour income share. However, the initial shock on public R&D is 70% running up to 100% and it leads to a fall of the labour share by a maximum of 4%. A shock of 70% for a given GDP means that, for example, public R&D goes from 1% of GDP to 1.7% of GDP. This is completely unrealistic. Even a 7% increase is unrealistic and would lead to a transitional fall in the labour share by at most 0.4% in this estimate. As a matter of choice of variables, it is unclear why private R&D and TFP are not included. Also, the theoretical explanation of the result is doubtful. There is an ad valorem subsidy on the selling price for the adopter of intermediates which works like a reduction of intermediate prices, leading to a boost in using intermediates. This has an effect on efficient labour because, without any justification, intermediates boost efficient labour supply by assumption. An increase in efficient labour supply, under a plausible elasticity of capital-labour substitution of 0.4, leads to a fall in wages by 2.5 times the percentage increase of labour supply. Moreover, it is unclear why subsidizing adoption of intermediates should be considered as public R&D. In standard public finance terminology, it is a subsidy for investments, not R&D.

[5] Interaction of R&D variables with time trends in TFP regressions of Guellec and van Pottelsberghe de la Potterie (2004) comes close though. Marchese and Privileggi (2020) have asymptotically constant unit costs but in the transition to a balanced growth path they may change.

R&D stock variables and a time trend, which has a negative regression coefficient interpreted as a trend in the production function for TFP of about -0.3%, but weaker for Japan, Germany and the USA.[6] Porter and Stern (2000) find negative trend in patent production functions when lagged dependent variables are taken into account; they interpret them as effects of international trade and changes in property rights. Abdih and Joutz (2006) find a negative trend in production of patent applications, which they interpret as detrending of variables for a number of researchers growing more quickly than that of patents. Link and Scott (2019) start from a production function for knowledge output, new scientific publications, Q = AF(K, L). The expenditure for the factors is rK+wL = I. Estimating the growth rate of *A* from the function *Q = AF(K, L)* they find a negative growth rate for *A*.[7] This approach implicitly captures also experience from former expenditure in *A*. We accumulate the expenditures on four types of R&D and use the four stocks as experience indicator. In the knowledge VES function used below, factor-augmenting technical change appears together with the experience indicators. We also find negative R&D- augmenting technical change in the knowledge production function by way of comparison of the marginal productivities of VES functions with the VECM estimates of Soete et al. (2020b).

The methodology of comparison of VES based theory with data driven VECM results goes together with the finding that domestic and foreign private and public R&D stocks grow more quickly than TFP measures. This latter result is often attributed to Bloom et al. (forthcoming), but it is obtained without distinguishing between private and public R&D. (Soete et al. 2020a, b) show that this is directly visible from the VECMs for Canada, France, Netherlands and Spain, which have no differenced terms in the VECMs and the constants provide the long-term growth rates.[8] However, they also show that internal rates of return for additional public R&D are still very high. Therefore, we hesitate to speak of a crisis in the TFP-R&D relation.

In section 2, we provide an economic growth model that links productivity and four R&D variables to preferences. In section 3, a semi-endogenous growth model shows growth driven by foreign R&D for a Cobb-Douglas function for productivity development; the relation of

[6] We cannot exclude the possibility that the effect is only one from detrending the variables.

[7] Related papers often ignore a time trend in log-level versions corresponding to the constant in the differenced version. Therefore, they cannot test for negative or positive exogenous technical progress. For this reason, we do not deal with this literature extensively. The time trend is needed for detrending of the variables, not only as explanatory variable. Therefore, trends do not necessarily have to be statistically significant.

[8] These are cases without differenced lags in the VECM. The constant of the model then represents the long run growth rates. For other countries it is more circumstantial to see such results.

the CD model to VEC models requires unit coefficients though. In section 4, we use a VES function and relate it to VEC models showing that the VES model can be expected to be more in the line with the evidence from VECMs. Section 5 shows that balanced growth paths (same growth rates for several variables) exist for VES functions only under restrictions for the domestic parameters equivalent to CES functions. Section 6 deals with stability considerations in a descriptive and informal way. Section 7 compares the VES model with evidence from VECM estimates in order to find values for the VES parameters, showing that VES functions are realistic and balanced growth is not. Section 8 summarizes and concludes.

2. A Basic Model

2.1 Basics

The output production functions for the home and foreign (indicated by a '*') country, ignoring capital and labour, are

$$Q = A, Q^* = A^* \quad (1)$$

The productivity functions are

$$\mathrm{A} = F(A_{-1}, R, t) \qquad \text{and} \qquad A^* = F^*(A^*_{-1}, R^*, t) \quad (2)$$

*R, R** are (1, 4) vectors of R&D stock elements $R_j, R_j^*, j = b, g$ where *b* and *g* indicate business and non-business or (semi-) government R&D. The function may be shifted by exogenous technical change indicated by time *t*. Analogous to the ideas of 'learning by doing' (Arrow 1962), 'perspectives of experience' (Boston Consulting Group 1968), and 'learning from watching' (King and Robson 1993) we consider R&D capital stocks as research experience indicators. To keep the model simple, we assume that spillovers stem from R&D, for example through patenting including the publication of the innovation, and not from productivity, which includes complex phenomena such as organization of the firms and other microeconomic and institutional issues, which are beyond the scope of this macroeconomic paper. R&D capital builds up according to the perpetual inventory method in the empirical literature (suppressing the index for current time),

$$R_{j,t+1} = R_j + I_j - \delta_j R_j, \; j = b, g \quad (3)$$

Again, we assume a similar equation for a foreign country. Investment, I_j, includes the minimized costs for human capital, labs and low- or medium skill support of researchers. The advantage of this approach is that we do not have to model all of these factors separately.[9] The last term represents depreciation in the learning indicator. The disadvantage is that we cannot distinguish quantities and prices of factors in this simple version of the model. Doing so would lead us to larger models, which go beyond the limited number of variables typically included in vector-error-correction models to which we want to relate our model. A share *s* of output goes into R&D investment

$$I_j = s_j Q, j = b, g \tag{4}$$

2.2 Dynamics and stability

With the production function (1) in the investment equation (4), and the result in the dynamic equations in (3) we get

$$R_{b,t+1} = R_b + s_b F(A_{t-1}, R_b, R_g; R_b^*, R_g^*, t) - \delta_b R_b,$$

$$R_{g,t+1} = R_g + s_g F(A_{t-1}, R_b, R_g; R_b^*, R_g^*, t) - \delta_g R_g \tag{3'}$$

Foreign R&D terms are exogenous for the home country by assumption. Subtracting current R-terms on both sides and dividing by them results in the growth rate version of these equations. Subtracting also $g \equiv \frac{A_{+1}}{A} - 1$ yields

$$g_b - g \equiv \frac{R_{b,t+1} - R_b}{R_b} - g = s_b F(A_{-1}, R_b, R_g; R_b^*, R_g^*, t)/R_b - \delta_b - g,$$

$$g_g - g \equiv \frac{R_{g,t+1} - R_g}{R_g} - g_A = s_g F(A_{-1}, R_b, R_g; R_b^*, R_g^*, t)/R_g - \delta_g - g \tag{3''}$$

Constant A/R_i ratios can insure constant growth rates. This requires that A and R_i have the same growth rate in the long run, and that this is compatible with the specification of the production functions used for F below. We define $y_g \equiv R_g/A$, $y_b \equiv R_b/A$. With these definitions, (3‘‘) can be re-written as follows.

[9] It is possible though to model the underlying cost-minimization in relation to the productivity production function of endogenous growth theory as well as that of Link and Scott (2019).

$$\hat{y}_b = \frac{s_b}{y_b} - \delta_b - g$$

$$\hat{y}_g = \frac{s_g}{y_g} - \delta_g - g \qquad (3''')$$

Multiplying the first equation by y_b and the second by y_g we get

$$\dot{y}_b = s_b - (\delta_b + g)y_b,\ \dot{y}_g = s_g - (\delta_g + g)y_g \qquad (3^{iv})$$

The steady-state solution for y_g and y_b is

$$y_b^s = \frac{s_b}{(\delta_b + g)}, \quad y_g^s = \frac{s_g}{(\delta_g + g)} \qquad (5)$$

For positive (negative) growth rates, both sides in (3^{iv}) are positive (negative) and therefore y-terms are increasing (decreasing) and thereby decrease (increase) the right-hand side. The process is therefore stable, going to the steady state solution (5) if a constant growth rate g exists (even if it varies in the adjustment process), which depends also on the production functions assumed below. As a result, R_g, A, and R_b have the same growth rate, if the production functions relating them do not contradict this partial stability analysis.

2.3 *Extension to endogenous savings*

We assume that governments' search for an optimal mix of policy instruments – in practice including evaluation studies surveyed in Ziesemer (2019) - can be approximated by a utility maximizing choice of public R&D, and that firms try the same.[10] The country's problem at time t in finding the optimal savings ratios is to maximize utility from consumption subject to the dynamic R&D equations:

$$Max_{s_b, s_g} \sum_{\tau=t}^{T} \rho^{\tau-t} U\big[(1 - s_{b,\tau} - s_{g,\tau})F(A_{\tau-1}, R_\tau, \tau)\big]$$

$$\text{s.t.} \quad R_{b,\tau+1} = R_{b,\tau} + s_{b,\tau}F(A_{\tau-1}, R_\tau, \tau) - \delta_b R_{b,\tau},$$

$$R_{g,\tau+1} = R_{g,\tau} + s_{g,\tau}F(A_{\tau-1}, R_\tau, \tau) - \delta_g R_{g,\tau},$$

given initial values for $A_{t-1} = A(t-1)$, $R_{b,t} = R_b(t)$ and $R_{g,t} = R_g(t)$, and discount factor $\rho < 1$.

[10] Park (1995) assumes that only private firms do this for private R&D.

T is the time horizon, τ is the index of future time periods with beginning value t as the present period. The constraints (3') include (1), (2) and (4) and use the definition $R_\tau = (R_{b,\tau}, R_{g,\tau}; R^*_{b,\tau}, R^*_{g,\tau})$. If T goes to infinity this is a standard dynamic optimization problem. If $T\text{-}t = 2$ we have a two period model. If $T\text{-}t = 1$ this is a myopic one-period model with no reason to invest unless we specify a salvage function.

We solve the dynamic R&D equations (3') for the savings ratio and insert them into the objective function. The result with $g_{j,\tau} = (R_{j,\tau+1} - R_{j,\tau})/R_{j,\tau}$ is

$$Max_{R_{b,\tau+1,}R_{g,\tau+1}} \sum_{\tau=t}^{T} \rho^{\tau-t} U\left[F(A_{\tau-1}, R_\tau, \tau) - (g_{b,\tau} + \delta_b)R_{b,\tau} - (g_{g,\tau} + \delta_g)R_{g,\tau}\right]$$

given initial values for $A_{t-1} = A(t-1)$, $R_{b,t} = R_b(t)$ and $R_{g,t} = R_g(t)$.

The first-order conditions for an interior solution are[11]

$$\rho U'_{\tau+1}\left[F_{2,\tau+1} - \delta_b\right] - U' = 0 \quad \text{or} \quad F_{2,\tau+1} - \delta_b = U'/(U'_{\tau+1}\rho) \tag{6a}$$

$$\rho U'_{\tau+1}\left[F_{3,\tau+1} - \delta_g\right] - U' = 0 \quad \text{or} \quad F_{3,\tau+1} - \delta_g = U'/(U'_{\tau+1}\rho) \tag{6b}$$

The suffix 2 or 3 denotes derivations with respect to the second or third argument in the function F, evaluated at period $\tau+1$. As control variables are substituted, concavity of the objective function w.r.t. $R_{b,\tau+1,}, R_{g,\tau+1}$ and their changes is a sufficient condition for optimality (Kamien and Schwartz 2001). As the function under the utility function is linear in the changes, even under linear utility concavity of F with respect to R-terms is sufficient; under strictly concave utility even mildly increasing returns of F could be allow for.

The net marginal products should be equal to each other,[12] and, as usual, marginal products should equal capital costs. For an iso-elastic specification of the utility function

$$U(c) = \frac{c^{1-\omega}}{1-\omega}, \text{ we get } \frac{U'}{(U'_{\tau+1}\rho)} = \frac{c^{-\omega}}{c_{\tau+1}^{-\omega}\rho} = \frac{(1+g_c)^\omega}{\rho}.$$

It follows from this result that the right-hand sides of (6a) and (6b) are constant if the growth rate of consumption, $c = F(A_{\tau-1}, R_\tau, \tau) - (g_{b,\tau} + \delta_b)R_{b,\tau} - (g_{g,\tau} + \delta_g)R_{g,\tau}$, is constant. This is the case for constant and identical growth rates of R-terms and F *(sufficient)*.

[11] The (-1) before U' on the left-hand side comes from deriving $g_{j,\tau} = (R_{j,\tau+1} - R_{j,\tau})/\, R_{j,\tau}$ with respect to $R_{j,\tau+1}$, which results in $1/R_{j,\tau}$, which is multiplied by the R-term multiplied to g in the objective function. Moreover, deriving gR for the next period makes g-terms drop out.

[12] If the sum of savings ratios is exogenous, we could maximize productivity through the manner of splitting the sum. Then, equal marginal products would also be a requirement for the maximization.

Convergence of the integral to a constant value is necessary for a maximum to exist. It requires that the growth rate of discounted utility is negative, which is the case if the discount rate dominates the growth rate of utility in the long run, $log\rho + (1-\omega)g < 0$, or $(1-\omega)g < -log\rho$.[13]

3. Semi-endogenous growth with Cobb-Douglas function, and exogenous foreign R&D capital

3.1 Steady-state definition and growth rate in the Cobb-Douglas case

A special case of the function (2) for A is the Cobb-Douglas function with $1 > \alpha, \beta > 0; C \geq 1$, and unspecified signs for the exponents of the lagged dependent variable, the foreign variables and exogenous technical change, $\varphi, \gamma, \mu, b \lesseqgtr 0$,

$$A = (A_{-1})^{\varphi} R_b^{\alpha} R_g^{\beta} R_b^{*\gamma} R_g^{*\mu} C e^{bt} \tag{2'}$$

C is a constant to adjust the value level of the right-hand side to the requirements of the data on the left-hand side. As Kaldor and Mirrlees (1962), we add exogenous growth to the endogenous arguments. The empirical literature interprets $\gamma, \mu > 0$ as dominance of positive spillovers, and $\gamma, \mu < 0$ as dominance of negative spillovers and competition effects from abroad (Luintel and Khan 2004). Of course, in principle, γ and μ could have opposite signs, for example positive spillovers from public R&D and dominant competition effects from private foreign R&D, $\gamma < 0, \mu > 0$. Taking growth rates, and assuming equality of growth rates for A and domestic R-terms we get

$$\hat{A} = \frac{\gamma \widehat{R_b^*} + \mu \widehat{R_g^*} + b}{1 - \varphi - \alpha - \beta} \equiv g \tag{2''}$$

This result shows semi-endogenous growth, as A is a function of endogenous variables, but the growth result has exogenous variables on its right-hand side. If $b >(<) 0$, it adds an element of (negative) exogenous growth. So far, we assume $1 - \varphi - \alpha - \beta > 0$; foreign R&D growth should be stronger than a negative trend in order to get a positive productivity growth rate. Park (1995), Eaton and Kortum (1997), NESTI (2017), and Ahmed and Bhatti (2020) emphasize this strong role of foreign R&D for small countries;

[13] The growth rate of the discount factor $\rho^{\tau-t}$, calculated as log-difference, is $log\rho < 0$.

Whereas semi-endogenous growth models normally have the population growth rate on the right-hand side, here foreign R&D has this role. By implication, growth stops if foreign R&D would stop growing in case $\gamma, \mu > 0$ and the exogenous growth rate b is zero.[14] Even if b is positive, $\gamma, \mu < 0$ would imply that foreign R&D can have a competition effect, for example making the more productive domestic sectors smaller and thereby make productivity growth negative. In this new perspective it is therefore possible that productivity growth has phases of being negative, not only in recessions via the degree of capacity utilization (see Penn World Tables). However, the strong role of population growth may come back in future research if capital and labour would be re-introduced and foreign variables get endogenous.

3.2 Endogenous savings ratio for the Cobb-Douglas function

With a Cobb-Douglas production function like (2') we get $F_2 = \alpha A/R_b$, and $F_3 = \beta A/R_g$ from (6a) and (6b). Replacing the marginal products in (6a) and (6b) yields solutions of R&D-stock/productivity ratios as functions of the consumption growth rates or vice versa:

$$F_{2,\tau+1} = \alpha A/R_b = \frac{(1+g_c)^{\omega}}{\rho} + \delta_b \qquad \text{or} \qquad \frac{R_b}{A} = \frac{\alpha}{\frac{(1+g_c)^{\omega}}{\rho} + \delta_b} \tag{7a}$$

$$F_{3,\tau+1} = \beta A/R_g = \frac{(1+g_c)^{\omega}}{\rho} + \delta_g \qquad \text{or} \qquad \frac{R_g}{A} = \frac{\beta}{\frac{(1+g_c)^{\omega}}{\rho} + \delta_g} \tag{7b}$$

Using these stock ratios in (5) with the definition of $y_j \equiv R_j/A$ and yields endogenous steady-state savings ratios, with g from (2''):

$$s_b = y_b^s(\delta_b + g) = \frac{\alpha(\delta_b + g)}{\frac{(1+g_c)^{\omega}}{\rho} + \delta_b}\ , \qquad s_g = y_g^s(\delta_g + g) = \frac{\beta(\delta_g + g)}{\frac{(1+g_c)^{\omega}}{\rho} + \delta_g} \tag{8}$$

As $(1+g)^{\omega}/\rho > 1$ for $\omega > 0$, and all other expressions are below unity, the savings ratios are below unity. In a steady state with constant growth rates for consumption and productivity, this determines the savings ratios. This result is only valid for productivity production functions such as the Cobb-Douglas function for which (7a,b) determine the *A/R* ratios, and in the steady state for which consumption *c* has the same growth rate as *A*, R_b, and R_g, because these assumptions have been used in the derivation of (2'').

[14] Cova et al (2017) use this function for φ=b=0, with slightly different R&D variables.

3.3 Existence and stability of a steady state with Cobb-Douglas productivity function

The Cobb-Douglas function above is one possibility that may allow for a stable steady state, and it delivers the growth rate of *A*. We define $R^* \equiv R_b^{*\gamma} R_g^{*\mu} e^{bt}$, $y_g \equiv R_g/A$, $y_b \equiv R_b/A$. From (3’’’), we then get

$$\hat{y}_b = s_b (A_{-1})^{\varphi} R_b^{\alpha-1} R_g^{\beta} R^* C - \delta_b - g$$

$$\hat{y}_g = s_g (A_{-1})^{\varphi} R_b^{\alpha} R_g^{\beta-1} R^* C - \delta_g - g$$

with *g* from (2’’). Multiplying through by *y*-terms then delivers (3^{iv}) again. Existence and stability are ensured for y terms, given *g*. The left-hand side equal to zero requires that domestic *R* and *A* terms have the same growth rate. Existence and stability of a steady state then requires that the Cobb-Douglas terms are constant and go to the growth rate of *R* and *A* terms. The production function (2’), using the definition of $R*$, in terms of *y* expressions is

$$A^{1-\alpha-\beta} = (A_{-1})^{\varphi} y_b^{\alpha} y_g^{\beta} R^* C$$

Taking growth rates yields

$$\hat{A} = \frac{1}{(1-\alpha-\beta)} (\varphi \widehat{A_{-1}} + \alpha \widehat{y_b} + \beta \widehat{y_g} + \widehat{R^*}) \qquad (2''')$$

According to (3^{iv}), *y* terms converge to constant values and therefore their growth rates go to zero.

If $\frac{\varphi}{(1-\alpha-\beta)} < 1$, (2’’’) is a stable process going to

$$\hat{A} = \widehat{A_{-1}} = \frac{\widehat{R^*}}{1-\varphi-\alpha-\beta} \quad , \qquad (2^{iv})$$

This is (2’’) again. Therefore, this proves existence and partial stability for y terms and the growth rate of A without further restrictions on the parameters of the Cobb-Douglas function. The stability analysis is only partial because the feedback effects from the growth of *A* to (3^{iv}) and vice versa are not yet considered. A special case of this is fishing out $-\varphi = \alpha + \beta$ requiring a special negative value of φ, making the denominator of (2^{iv}) equal to unity . Below we will consider other functions.

For the optimization case, things are only slightly more complicated. In the optimization, we have substituted the savings ratios from the dynamic equations into the objective function

$$s_{b,\tau} = \frac{(g_{b,\tau} + \delta_b)R_{b,\tau}}{F(A_{\tau-1}, R_\tau, \tau)} = (g_{b,\tau} + \delta_b)y_{b,\tau}$$

$$s_{g,\tau} = \frac{(g_{g,\tau} + \delta_g)R_{g,\tau}}{F(A_{\tau-1}, R_\tau, \tau)} = (g_{g,\tau} + \delta_g)y_{g,\tau}$$

This corresponds to (5) in terms of non-steady-state values. Replacing the savings ratios in (3^{iv}) yields

$$\dot{y}_{b,\tau} = (g_{b,\tau} + \delta_b)y_{b,\tau} - (\delta_b + g)y_{b,\tau} = (g_{b,\tau} - g)y_{b,\tau}, \qquad (3^{v}a)$$

$$\dot{y}_{g,\tau} = (g_{g,\tau} + \delta_g)y_{g,\tau} - (\delta_g + g)y_{g,\tau} = (g_{g,\tau} - g)y_{g,\tau} \qquad (3^{v}b)$$

There is an additional effect on the savings ratios now. The optimality conditions (7a) and (7b) are hard to fulfill with equality, because *A/R*-terms are given at any moment in time because productivity and R&D stocks move only slowly. This would suggest that the growth rate of consumption adjusts. However, it cannot balance two equations. Whenever the marginal product of private or public R&D is above (below) its optimum value, this is the case because its R&D level is below (above) optimum and therefore its saving ratio should be higher (lower)[15] than in the steady state. Therefore, temporarily higher (lower) growth rates of private or public R&D lead to a speed up of the respective *R/A* ratio. This is the message of (3^{v}a, b). The growth rate difference is a policy reaction. The growth rate difference is set to zero if stability leads to a steady state where all growth rates are the same and the marginal productivity conditions hold.

3.4 A translation to the long-term relations of a VECM for the Cobb-Douglas function relations

The marginal productivity condition in equation (7a) suggests $logR_b = logA - log(F_2/\alpha)$, and equation (7b) suggests $logR_g = logA - log(F_3/\beta)$, where the last terms are a constant according to (6a, b) if consumption growth has reached its steady-state value. Combining the last two

[15] Actually, it must become zero, because the problem requires ‘a most rapid approach to a singular solution’, where the latter is the equality of the two marginal products. With no control variable in the equation, this requires extending only the relative low variable (Kamien and Schwartz 2001).

equations by elimination of *logA* leads to $logR_b = logR_g + log(F_3/\beta) - log(F_2/\alpha)$. Two of these or the Cobb-Douglas production function, could be long-term relationships in a cointegrated VAR in line with theoretical modelling.[16] This would lead to four possible cases with due adjustment of lags:

(i) The limiting case of a vector-error-correction model with full rank and estimation in levels with foreign terms as (weakly) exogenous, if confirmed by tests.

(ii) In case of no cointegration, the same model would be re-written in first differences.

(iii) In case of one cointegrating equation, we would have the following long-term relation from the productivity production function as expected value:

$$[(\varphi - 1)\log(\frac{A}{1+g}) + \alpha logR_b + \beta logR_g + \gamma logR_b^* + \mu logR_g^* + bt + c - g]_{-1} = 0$$

(iv) In case of two long-term relations, we would imagine having (with *u* and *v* as residuals)

$$E(u_{-1}) = 0 = \left[logA - logR_b - \log\left(\frac{F_2}{\alpha}\right)\right]_{-1},$$

$$E(v_{-1}) = 0 = [logR_b - \ logR_g - log\left(\frac{F_3}{\beta}\right) + log(\frac{F_2}{\alpha})]_{-1}$$

In (iii), we have replaced *A(-1)* by its steady-state value *A/(1+g)*. At the end of the bracket term, we have denoted by the sub-index (-1) that it enters with a lag in a VECM. Inside the brackets, it is necessary to include $g = logA - log(A(-1)$ in order to make the long-term relation identical to the log version of the production function divided by *A(-1)* on both sides. The long-term relation would have to be zero in a long-run equilibrium and therefore it is written in its homogenous form. A non-zero value represents a disequilibrium.

In (iv), public R&D drives private R&D according to the second equation, and, in turn, private R&D drives productivity according to the first equation. The marginal product terms could be replaced by the right-hand side of (7a) and (7b) if they hold with equality. Time trends do not appear in this case. Several terms here have unit coefficients for the case of a

[16] Osiewalski et al. (2020), in a similar approach, compare the Cobb-Douglas function for value added, capital and labour, to results from Bayesian cointegration analysis. They do find only one cointegrating equation and do not consider marginal products. Their results do not support the CD function. This is in line with literature favouring elasticities of substitution lower than unity. Antras (2004) uses marginal productivity conditions to estimate the CES parameter and finds a range below 0.9 for the elasticity of substitution.

Cobb-Douglas function. This may seem unrealistic at first sight, but unit coefficients feature prominently in the long-term relations of textbook examples. The consumption-income-investment relation (Lütkepohl 2005) and the liquidity-interest relationships (Juselius 2006) also have unit coefficients, which estimations only obtain approximately in the more flexible VECM approaches. Pesaran (1997) points out that only under special cases of functional forms we can link the VECMs directly to theoretical models. VECM estimation in a log-log approach could test whether the relations are similar to those obtained here for Cobb-Douglas functions. As this turns out to be dis-appointing (see Appendix VECMs), it may be interesting to look at other production functions as we do in the next section.

We may also use, in addition, output production function (1) suggesting *logQ = log (A)* as a long-term relation. Alternatively, this function could be used in one way or other to replace *log(A)*-terms above. Long-term relations in VECMs have two-way causality. The marginal productivity conditions would suggest, that given *A*, R-terms are determined or explained. Conversely, the three variables determine each other and R-terms drive *A* according to the productions function requiring adjustments of R-terms again. In sum, in principal we have a theoretical underpinning for the VECM work of Soete et al. (2020a,b) but the Cobb-Douglas functions are quantitatively unrealistic.

4. Mukerji VES and long-term relations

Suppose now that we use a VES (variable elasticity of substitution) function of Mukerji (1963) for the generation of productivity *A,* with *C* as a constant to determine the level of *A* in line with the data:

$$A = \left[a(A_{-1})^{\varphi} + he^{\alpha bt}R_b^{\alpha} + (1-h)e^{\beta xt}R_g^{\beta} + dR_b^{*\gamma} + fR_g^{*\mu}\right]^{1/\epsilon} C \equiv B^{1/\epsilon}C \qquad (2^{\mathrm{v}})$$

Then the marginal product of private and public R&D are

$$F_2 = \frac{1}{\epsilon} A^{1-\epsilon} e^{\alpha bt} h\alpha {R_b}^{\alpha-1} C^{\epsilon} \text{ and } F_3 = \frac{1}{\epsilon} A^{1-\epsilon} e^{\beta xt}(1-h)\beta {R_g}^{\beta-1} C^{\epsilon} \qquad (9)$$

Mukerji (1963) considers it as a generalization of the standard CES function. Arrow and Hurwicz (1958, p.550) and Houthakker (1960) use this function with $\epsilon = 1, b = x = 0$ for utility. They all do not use exponential trends or a lagged dependent variable. The terms in brackets, abbreviated as *B* here, are called 'addilog' in Houthakker (1960). For a positive effect of private R&D and its rate of technical change, b, we need either positive α, h, and ϵ, or two of them could be negative.[17] For a positive effect of public R&D and its rate of technical change, x, we need either positive β, $1\text{-}h$, and ϵ, or two of them could be negative. If all the exponential parameters in Greek letters have the same value, we have a CES function. If that value were zero, we would have a Cobb-Douglas function. We follow the convention for CES functions, to impose $1\text{-}h$ as the linear parameter of the second input. In the limiting case of a Cobb-Douglas function, h and $(1\text{-}h)$ are the cost shares of private and public R&D in productivity development. This assumption will help us later to find values for some of the parameters and show that private and public R&D have positive effects on the productivity A. Whereas these marginal productivity equations are log linear, the production function is not log-linear.[18] Taking logs of the marginal products in (9) yields long-term relations for A and R_b or R_g in the form of (9a) and (9b) below, as well as for R_b and R_g in the form of (9c):

$$logA = \frac{(1-\alpha)}{(1-\epsilon)} logR_b - \frac{\alpha b}{(1-\epsilon)} t + \frac{1}{(1-\epsilon)} \log\left(\frac{\epsilon F_2}{h\alpha C^{\epsilon}}\right) \tag{9a}$$

$$logA = \frac{(1-\beta)}{(1-\epsilon)} logR_g - \frac{\beta x}{(1-\epsilon)} t + \frac{1}{(1-\epsilon)} \log\left(\frac{\epsilon F_3}{(1-h)\beta C^{\epsilon}}\right) \tag{9b}$$

$$logR_b = \frac{(1-\beta)}{(1-\alpha)} logR_g + \frac{(\alpha b-\beta x)}{(1-\alpha)} t + \frac{1}{(1-\alpha)} log\frac{F_3}{F_2} + \frac{1}{(1-\alpha)} log\frac{h\alpha}{(1-h)\beta} \tag{9c}$$

We obtain the third by equating (9a) and (9b). In the first two equations, the second term is a time trend. For a constant growth rate of consumption, we have constant marginal products from (6a, b), if the equations hold with equality. Therefore, the third term is a constant. These terms consist of the parameters of the production function (2^{v}) and the conditions (6a, b) in case of equality, unless the growth rate is taken as a variable too. The third equation is a long-term relation between R_b and R_g with time trend, with a positive or negative coefficient. These relations appear not only in the model but also in the corresponding long-term relations in vector-error-correction models in Soete et al. (2020b), where all relations may have two-way

[17] In a previous version, we formulated the text using the assumption of positive parameters. This implies high elasticities of substitution. For example in the special case of a CES function, $Y = C[hx^{\rho} + (1-h)z^{\rho}]^{1/\rho}$ the elasticity of substitution is $\sigma = 1/(1-\rho)$, which is larger than unity if $\rho > 0$, but smaller than unity if $0 > \rho > -1$. Therefore, we do not want to exclude the cases with negative parameters.

[18] The version on the right-hand side of (9) is log-linear in B. In principle, we could estimate B as a non-linear part of the production function.

causality.[19] If we would assume only a neutral rate of technical change there would be no time trend in equation (9c).

Looking at the equations in detail, we see in (9a) that either $\alpha < 1, \epsilon < 1 \; or \; \alpha > 1, \epsilon > 1$ implies for finite parameter values that private R&D has a positive effect on productivity. In the last term, the existence of the log expression requires that the expression under the log is positive; either all terms are positive or an even number of them must have a negative sign.

For (9c) the following holds. Either $\alpha < 1, \beta < 1 \; or \; \alpha > 1, \beta > 1$ implies for finite parameter values that the slope is positive, and the R&D variables are complements. If these conditions are not fulfilled, they are substitutes.[20] Mukerji (1963) shows that $\frac{(1-\beta)}{(1-\alpha)}$ is the ratio of the elasticities of substitution vis-a-vis other factors of production. If marginal product and depreciation rates in (6a, b) are equal, the first part of the constant is zero; in this case, $h\alpha$ and $(1-h)\beta$ must have the same sign for the last log expression to exist. If $h=1-h$ and $\alpha = \beta$, under equal rates of depreciation, both parts of the intercept were zero. If $\alpha b = \beta x$, the time trend would also have no effect, and in general it can have any sign. Private R&D could exist without public R&D only if the constant or the coefficient of the time trend are positive. However, the marginal product of public R&D would go to infinity if public R&D goes towards zero, provided $\beta - 1 < 0$.

Equations (9a) and (9c) have six coefficients for the slopes of the R&D variables, the time trends and the constants. They consist of more than six parameters. However, if we assume $b = x$, and $F_2/C^{\epsilon} = F_3/C^{\epsilon} = \theta$ the slopes of R&D variables and time trends as well as the constants are determined by $b, \alpha, \beta, \epsilon, \theta, and \; h.$ Conversely, if we have estimates of slopes and constants, we can back out the values of the parameters. The estimated coefficients then generate six equations from which we could try to solve for the six parameters with calibrating assumptions. We can then compare these parameters to those of the Cobb-Douglas function or more general CES functions.

[19] Without imposing the assumption of a long-term relation, the marginal productivity conditions could be estimated also using the growth rate of consumption as a variable.

[20] Definitions of static substitutes and complements based on (9c) can deviate from definitions of dynamic complements or substitutes using the total system effect of a VECM as in Soete et al. (2020b).

5. ***Steady state growth for two countries***

5.1 Conditions for a balanced growth path with Mukerji VES functions

Narrowing down the consideration to explore the possibility of balanced growth results of the growth model leads to the following. Taking differences of equation (9a) and assuming that productivity and *efficient* private R&D, $e^{bt}R_b$, have the same growth rate, we get

$$g = dlogA = \frac{(1-\alpha)}{(1-\epsilon)} dlogR_b - \frac{\alpha b}{(1-\epsilon)} = dlogR_b + b.$$

Solving this for the growth rate of private R&D yields

$$dlogR_b = \frac{b+\frac{\alpha b}{(1-\epsilon)}}{\frac{(1-\alpha)}{(1-\epsilon)}-1} = \frac{b(1-\epsilon)+\alpha b}{\epsilon-\alpha} = b\,\frac{1-\epsilon+\alpha}{\epsilon-\alpha}$$

Solving for the productivity growth rate yields

$$g = dlogA = dlogR_b + b = b\,\frac{(1-\epsilon)+\alpha}{\epsilon-\alpha} + b = b\,\frac{(1-\epsilon)+\alpha}{\epsilon-\alpha} + \frac{b(\epsilon-\alpha)}{\epsilon-\alpha} = \frac{b}{\epsilon-\alpha} \quad (10a)$$

The interpretation is that we have semi-endogenous growth, ultimately driven by exogenous technical change if the assumption of balanced growth of productivity and efficient private R&D holds.

The second equation, (9b), shows a relation between productivity and public R&D. Equal growth rates of productivity and efficient public R&D, $e^{xt}R_g$, correspondingly lead to

$$g = dlogA = \frac{x}{\epsilon-\beta}, dlogR_b = x\,\frac{(1-\epsilon)+\beta}{\epsilon-\beta} \quad (10b)$$

A balanced growth path for these variables therefore requires $b(\epsilon-\beta) = x\,(\epsilon-\alpha)$, which holds if *$\beta = \alpha$ and $b = x$ (sufficient)*. The sufficient conditions for a balanced growth path are a step back towards a CES function from the given VES function. Only if public and private R&D are equally productive in the production of *A* and have identical rates of technical progress can we have a steady state with identical growth rates for productivity and efficient R&D variables.

For a set of sufficient conditions for the existence of a steady state, suppose

(i) for the foreign country there is a VES function like (2^{v}) with variables and parameters having a '*',

(ii) there are relations symmetric to (10a, b), $g^* = \frac{b^*}{\epsilon^* - \alpha^*} = \frac{x^*}{\epsilon^* - \beta^*}, \alpha^* = \beta^*, b^* = x^*$ (11)

(iii) $\alpha = \beta = \varphi = \epsilon, \gamma = \mu$, and for the foreign country $\alpha^* = \beta^* = \epsilon^* = \varphi^*, \gamma^* = \mu^*$.

(iv) $\epsilon g = \gamma(g^* - b^*), \epsilon^* g^* = \gamma^*(g - b)$ or $\frac{g}{g^* - b^*} = \gamma/\epsilon, \frac{g^*}{g - b} = \epsilon^*/\gamma^*$.

Property (ii) for the foreign country follows from the previous lines. Equal growth rates of all terms on the right-hand side of the VES function (2^{v}) require

$$\varphi g = \alpha g = \beta g = \gamma(g^* - b^*) = \mu(g^* - b^*)$$

From this equation, we find property (iii) using the $\beta = \alpha = \phi$. The left-hand side must have a corresponding growth rate[21] and therefore balanced growth requires $\beta = \alpha = \phi = \epsilon$, and correspondingly for the foreign country $\beta^* = \alpha^* = \epsilon^* = \varphi^*$. The second part of property (iii) follows from the equality of growth rates for foreign terms in (2^{v}). Property (iii) implies that these constraints for a balanced growth path are restrictive in relation to the VES function.

Property (iv) follows from the requirement of equal growth rates for domestic and foreign terms in (2^{v}). As all other parameters have already been used, assumption (iv) can be seen as putting a constraint on γ and γ^* as necessary for balanced productivity growth of a two-country model. With these assumptions, arguments in the VES function (2^{iv}) have exponential growth functions with identical exponents multiplied to *t*, and we can re-write the production function, together with properties (i)-(iv) for the home country as

$$A = e^{\alpha g t/\epsilon}\left[\underline{B}\right]^{1/\epsilon} C = e^{\gamma(g^* - b^*)t/\epsilon}\left[\underline{B}\right]^{1/\epsilon} C \tag{12}$$

Underlining of *B* expresses that all arguments in *B,* multiplied to the growth rate term e^{gt} or identical ones according to (iv) above, are fixed to a certain value when hypothetically arriving at the steady state.

5.2 Dynamics with VES function and balanced growth constraints

Finding steady state conditions can be slightly different from finding a balanced growth path because we need the whole model. Therefore, we do not use the results of the previous sub-

[21] For this purpose, we write (2^{v}) as $A^{\epsilon} = BC^{\epsilon}$

sections in the current one unless explicitly mentioned. We define $y_b \equiv e^{bt}R_b/A, y_g \equiv e^{xt}R_g/A$. From (3'), where these definitions were used for *b = x = 0*, with Mukerji-VES function, we then get the accumulation dynamics of the VES growth model:

$$\hat{y}_b = s_b\left[a(A_{-1})^{\varphi} + he^{\alpha bt}R_b^{\alpha}+(1-h)e^{\beta xt}R_g^{\beta}+dR_b^{*\gamma}+fR_g^{*\mu}\right]^{1/\epsilon}C/R_b - \delta_b - (g-b) \quad (13)$$

$$\hat{y}_g = s_g\left[a(A_{-1})^{\varphi} + he^{\alpha bt}R_b^{\alpha}+(1-h)e^{\beta xt}R_g^{\beta}+dR_b^{*\gamma}+fR_g^{*\mu}\right]^{1/\epsilon}C/R_g - \delta_g - (g-x) \quad (14)$$

Multiplying through by y-terms yields

$$\dot{y}_b = \frac{s_b\left[a(A_{-1})^{\varphi}+he^{\alpha bt}R_b^{\alpha}+(1-h)e^{\beta xt}R_g^{\beta}+dR_b^{*\gamma}+fR_g^{*\mu}\right]^{1/\epsilon}C}{Ae^{-bt}} - (\delta_b + g - b)y_b$$

(15)

$$\dot{y}_g = \frac{s_g\left[a(A_{-1})^{\varphi}+he^{\alpha bt}R_b^{\alpha}+(1-h)e^{\beta xt}R_g^{\beta}+dR_b^{*\gamma}+fR_g^{*\mu}\right]^{1/\epsilon}C}{Ae^{-xt}} - (\delta_g + g - x)y_g$$

(16)

In an appendix, we show that we get a steady state only for very special cases of parameter values. For the domestic dynamic equations this can be summarized as

either (i) b=x= 0, or (ii) ϵ = 0, or (iii) α = β = ϕ with ϵ = 1, or (iv) the CES case for domestic variables, α = β = ϕ = ϵ, in all four cases together with $b\frac{1-\epsilon+\alpha}{\epsilon-\alpha}\epsilon = \gamma(\frac{1-\epsilon^*+\alpha^*}{\epsilon^*-\alpha^*})b^*$ and b* = x* and γ = μ.

From the dynamic equations of the foreign countries, we would get by symmetry the conditions

either (i) b* = x* = 0, or (ii) ϵ* = 0, or, or (iii) ϵ* = 1 with α* = β* = ϕ*, or (iv) the CES case for foreign variables, α* = β* = ϕ* = ϵ*, in each of the four case together with $b\frac{1-\epsilon+\alpha}{\epsilon-\alpha}\gamma^* = \epsilon^*(\frac{1-\epsilon^*+\alpha^*}{\epsilon^*-\alpha^*})b^*$ and b = x and γ* = μ*.

This proves that the existence of a steady-state solution in the presence of a restricted Mukerji VES function and given savings ratios, optimal or not, is possible only under very restrictive conditions. Stability towards the stationary *y* terms for given *g-b, g-x* and *g*-b*, g*-x** at the end of (15) and (16) also works analogous to the analyses above. If *g-b, g-x or g*-b*, g*-x** would be higher than the fraction of their steady state levels, y-dot terms would be negative and slow down the growth of the efficient R/A ratios, which would reduce A growth. However, feedback effects would again be too complicated for qualitative analysis. Domestic and foreign growth rates can differ in the steady state of this model, but they must be proportional to each other. Otherwise, countries will have non-proportional growth rates and steady states do not exist, because the VES without restrictions contradicts the part of the system without production function.

In general, a VECM with a linear time trend or a growth model with VES functions are more flexible than a balanced growth or steady state model, and the long-run solution can have different growth rates for each variable. Balanced growth or steady states are unlikely to occur because they require very special parameter constellation. We provide some evidence for this below in section 7. Without parameter restrictions, the theoretical model would consist of equations (3') and (2^v) with exogenous foreign variables.

6. Existence and stability of attractors in theoretical and empirical growth models: A verbal comparison

Growth models like the one above, either in their theoretical forms or with calibrated and estimated parameters, are non-linear difference equation systems because they contain the production functions like (2^v). Empirical models as vector-autoregressive (VAR) models, in particular also vector-error-correction models, are log-linear models, which may contain the same variables as the theoretical models, and the long-term relations may be interpreted as marginal productivity conditions as in (9a,b), which may be log-linear even if the production functions like (2^v) are not. The first common property of dynamic theoretical and empirical models is that they are difference equation systems.

For difference equation systems, we can carry out a dynamic analysis. Details of this depend on the type of the system. For linear models, such as the VAR and VECM or theoretical models if they are log-linear, this is programed in the econometric packages. For non-linear models such as the basic neoclassical model of Solow (1956) with a CES function, there is only one difference equation and it may have stable steady states according to graphical analysis through the falling marginal product of capital.[22] If such models are extended to have two dynamic equations, analysis of flow diagrams can derive conditions for existence and

[22] For CES functions there may be exceptional constellation where a steady state does not exist (Burmeister and Dobell 1970).

stability of steady states (see Ziesemer 1990) or dynamics without steady states. For models that are more complex there may be stability theorems, but they often depend on special assumptions (Burmeister and Dobell 1970). If there are no suitable theorems for existence and stability, simulation can show where the systems go in the long run, either in terms of levels of variables or in terms of growth rates. In short, as theoretical and empirical models are difference or differential equation systems, we can analyze their dynamics with and without existence and stability of steady states. Instability of vector-error-correction models may lead to non-zero long-run impulse responses (Kilian and Lütkepohl 2017; see also Pesaran 2015). The second common property of theoretical and empirical models therefore is that we can carry out a dynamic analysis, which is related to a stability analysis if equilibria exist.

The stability analysis of theoretical models mostly starts with the definition of a steady state and then the analysis shows that the model goes to the steady state values, which are sometimes called attractors. For the models above, the common attractor are *A/R* variables. If stability for more complex models is carried out by way of simulations, programs can save the empirical values for all variables and the researchers can infer with the help of the model where the values of the variables go, either in terms of levels or in terms of growth rates. For the VECM in the variables of the model above, Soete et al. (2020a) show that the four variables have constant growth rates in the long run. However, but they also show that these growth rates may be different in the long run and therefore they do not form constant ratios. The VECM therefore is a bit more general from a numerical point of view than the stable cases of the theoretical model with special parameter values above, whereas the theoretical model indicates the causality reasoning in a slightly simplified way but in line with growth theory. Together these two aspects indicate that the results from VECMs are not mere data explorations. The third common property of theoretical and empirical models therefore is that we can find the attractors of the stable dynamic process, if they exist, which is the case for the theoretical VES model only under restrictive assumptions. If they do not exist, we can still have unbalanced, non-steady-state growth as in Soete et al. (2020a,b).

7. Some evidence for the parameter values

In this section, we try to get a first idea about the magnitude of the parameters in equations (9a-c). The coefficients of these equations consist of the parameters of the VES productivity production function. We compare the coefficients of these equations to some of the log-linear relations in the VECMs of Soete et al. (2020b). We select only those countries where the long-run relations consist of only two variables – Austria, Canada, France, Italy, Portugal. We present the details of the calculations in Appendix VECM. In order to find the values of parameters we have to assume that $x = b$, i.e. the same rate of exogenous factor-augmenting technical change for private and public R&D.[23]

The VES function (2^v), rewritten below Table 1 with $x = b$, shows that no single parameter alone does allow for an interpretation. All effects of private and public R&D and its factor augmenting technical change depend on the signs and values of several parameters shown in Table 1. With the exception of *(1-h)*, the linear parameter of public R&D, no row or column has only positive or only negative signs, nor are they close to zero as the exponents would be for the case of a Cobb-Douglas function. The VES parameter ϵ is negative for all countries. From the diversity of these results it follows that CES functions requiring $\alpha = \beta = \epsilon$ and balanced growth paths fulfilling the conditions of section 5 are not close to being relevant special cases. With the exception of *1-h*, the parameters do not have the same signs across the five countries. Therefore, we interpret them country by country now.

Table 1 Parameter values for the VES function and R&D effects on productivity

Country	b	ϵ	α	h	β	1- h	$\theta = F_{2,3}/C^{\epsilon}$	effect of R&D aug. TC (a)	effect of private R&D (b)	effect of public R&D (c)
Austria	-0.059	-2.95	-4.3	0.99999999999879	-0.137	> 0	2.147×10^{-18}	-	+	+
Canada	-0.037	-19	-4.3	2.5×10^{-806}	-202	<1	1.34×10^{-829}	-	+	+
France	0.035	-2.14	0.227	0.39	0.185	0.61	-4.93×10^{-6}	-	-	-
Italy	-0.036	-29	-13.9	2.25×10^{-12}	-17	<1	4.18×10^{-75}	-	+	+
Portugal	-0.089	-6.6	33.2	-2.2×10^{-73}	-0.36	>1	6.65×10^{-7}	-	+	+

Parameter estimates for $A = \left[a(A_{-1})^{\varphi} + he^{\alpha bt}R_b^{\alpha} + (1-h)e^{\beta bt}R_g^{\beta} + dR_b^{*\gamma} + fR_g^{*\mu}\right]^{1/\epsilon} C$. Legend: $\theta = F_{2,3}/C^{\epsilon}$ marginal product of R&D/productivity adjustment; (a) the sign of the productivity effect of R&D augmenting technical change is (sign of b) x (sign of α) x (sign of h) x (sign of ϵ); (b) sign of productivity effect of private R&D is (sign of α) x (sign of h) x (sign of ϵ); (c) sign of productivity effect of public R&D (sign of β) x (sign of h) x (sign of ϵ).

The three exponential parameters for Austria are negative. Considering these values and the positive values of the linear parameters, the effect of R&D augmenting technical change is negative. Private and public R&D both have positive effects on productivity A, working against the negative effect of exogenous R&D augmenting technical change. The equalized marginal products of R&D for Austria are also positive.

[23] Without this assumption other methods are needed to find the parameters.

The smallest absolute values for parameter estimates appear in the row for Canada. It seems plausible to say that the marginal products of R&D for Canada are close to zero even after productivity correction. The linear parameters for R&D are positive and the exponential ones are negative; together with the negative VES parameter ϵ there is a positive effect of private and public R&D. Also, for Canada we find a negative effect of R&D augmenting technical change because the exponential parameters are all negative, whereas the linear ones are positive. The growth of R&D is needed to keep productivity A away from a falling trend. The equalized marginal products are positive.

For France we find the only negative marginal products of R&D. The exponential and linear parameters of private and public R&D are positive but the VES parameter ϵ is negative and turns both R&D effects negative, in line with the negative marginal products of R&D. The same happens for R&D augmenting technical change. Soete et al. (2020b) formulate some doubts on the results for France. Therefore, it is plausible that they look problematic also here, from the perspective of a growth model.

Italy has positive linear parameters for R&D, but the one for private R&D is very low. The exponential R&D parameters are negative as is the VES parameter ϵ. Italy therefore has positive R&D effects, especially for public R&D, where also the exponential effect is slightly larger than for private R&D.

Portugal has opposite signs in both, linear and exponential parameters, and one positive and one negative for private and public R&D respectively. Together with the negative value of the VES parameter ϵ, this implies positive effects from additional private and public R&D, and negative effects from R&D augmenting technical change.

Comparing the parameters to the slopes of (9a) we see that private R&D has a positive relation with productivity except for Portugal, where the linear parameter in the production function is close to zero though; for the other countries the parameters α and ϵ are lower than unity. Public R&D and productivity have a positive relation according (9b) as β and ϵ are either larger or smaller than unity. Public and private R&D are complements except for Portugal, which is an exception to the rule that β and α are both above or both below unity. For all countries, we have $\beta - 1 < 0$, implying that the marginal product of public R&D goes to infinity if there were no public R&D. For private R&D Portugal is an exception.

All countries need growth of R&D in order to work against the negative effects of b.[24] These negative effects are unlikely to stem from mere detrending as in Abdih and Joutz (2006), because our VECM estimates are based on detrended variables and R&D augmenting technical change generates a trend in the theoretical model.

Some caution in accepting these results is in order because we have imposed the assumption that the rate of R&D augmenting technical change is the same for private and public R&D and the linear parameters are *h* and *1-h* respectively. However, most of the literature ignores this force all together.

8. Summary and conclusion

We have provided semi-endogenous growth models for productivity development with domestic and foreign public and private R&D and exogenous or endogenous savings ratios. Foreign R&D and exogenous technical change in the Cobb-Douglas case of a small country ultimately drive the long-run growth rate. If foreign R&D, which is driving the growth, is more damaging than exogenous technical change is helpful, the growth rate can be low and even negative.

Our growth model in combination with a CD or a restricted VES function can generate special cases of one or two long-term relations as in empirical VECM models.

The models of this paper provide the theoretical causality underlying the VECMs. Overall, the log-log regression approach for R&D in Soete et al. (2020a) is similar to the semi-endogenous growth models of this paper. This is the case especially for the marginal productivity conditions when moving from CD to VES function of the generalized CES type, where the marginal productivity functions are linear with more general coefficients then those from Cobb-Douglas functions; however, the production functions are not linear when going away from CD and standard CES functions.

The numerical results comparing the VECM with the VES in order to get the VES parameters deliver results for the production function with realistic effects for private and public R&D as well as exogenous factor-augmenting technical progress except for France.

[24] In contrast, the candidate for a growth rate for balanced growth, $\frac{b}{\epsilon-\alpha}$, is positive except for France, but balanced growth will not be achieved under the parameter values of Table 1.

In future research, we would like to get rid of simplifying assumptions made here: identical rates of factor augmenting technical change and linear parameters adding up to unity. This may require non-linear estimation of marginal productivity conditions together with the non-linear VES production functions.

Our model has suppressed the potential endogeneity of foreign R&D variables. In future work we will extend the theoretical model to have mutually optimal reactions of both countries to endogenous foreign R&D variables in differential games.

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Appendix: The dynamic system with Mukerji VES function

$$\dot{y}_b = \frac{s_b\left[a(A_{-1})^{\varphi} + he^{\alpha bt}R_b^{\alpha} + (1-h)e^{\beta xt}R_g^{\beta} + dR_b^{*\gamma} + fR_g^{*\mu}\right]^{1/\epsilon}C}{Ae^{-bt}} - (\delta_b + g - b)y_b \qquad (15)$$

$$\dot{y}_g = \frac{s_g\left[a(A_{-1})^{\varphi} + he^{\alpha bt}R_b^{\alpha} + (1-h)e^{\beta xt}R_g^{\beta} + dR_b^{*\gamma} + fR_g^{*\mu}\right]^{1/\epsilon}C}{Ae^{-xt}} - (\delta_g + g - x)y_g \qquad (16)$$

For a steady state with both sides equal to zero to exist, the large fraction has to be constant, because then the left-hand side can be zero and we can solve the equation for y_b, and y_g. Bringing the denominators into the brackets yields

$$\dot{y}_b = s_b\left[a\frac{(A_{-1})^{\varphi}}{(Ae^{-bt})^{\epsilon}} + h\frac{e^{\alpha bt}R_b^{\alpha}}{(Ae^{-bt})^{\epsilon}} + (1-h)\frac{e^{\beta xt}R_g^{\beta}}{(Ae^{-bt})^{\epsilon}} + d\frac{R_b^{*\gamma}}{(Ae^{-bt})^{\epsilon}} + f\frac{R_g^{*\mu}}{(Ae^{-bt})^{\epsilon}}\right]^{1/\epsilon} C - (\delta_b + g - b)y_b$$

$$\dot{y}_g = s_g\left[a\frac{(A_{-1})^{\varphi}}{(Ae^{-xt})^{\epsilon}} + h\frac{e^{\alpha bt}R_b^{\alpha}}{(Ae^{-xt})^{\epsilon}} + (1-h)\frac{e^{\beta xt}R_g^{\beta}}{(Ae^{-xt})^{\epsilon}} + d\frac{R_b^{*\gamma}}{(Ae^{-xt})^{\epsilon}} + f\frac{R_g^{*\mu}}{(Ae^{-xt})^{\epsilon}}\right]^{1/\epsilon} C - (\delta_g + g - x)y_g$$

Pulling the denominator terms under the exponents of the inputs results in

$$\dot{y}_b = s_b\left[a\left(\frac{A_{-1}}{(Ae^{-bt})^{\epsilon/\varphi}}\right)^{\varphi} + h\left(\frac{e^{bt}R_b}{(Ae^{-bt})^{\epsilon/\alpha}}\right)^{\alpha} + (1-h)\left(\frac{e^{xt}R_b}{(Ae^{-bt})^{\epsilon/\beta}}\right)^{\beta} + d\left(\frac{R_b^*}{(Ae^{-bt})^{\epsilon/\gamma}}\right)^{\gamma} + f\left(\frac{R_g^*}{(Ae^{-bt})^{\epsilon/\mu}}\right)^{\mu}\right]^{1/\epsilon} C - (\delta_b + g - b)y_b$$

$$\dot{y}_g = s_g\left[a\left(\frac{A_{-1}}{(Ae^{-xt})^{\epsilon/\varphi}}\right)^{\varphi} + h\left(\frac{e^{bt}R_b}{(Ae^{-xt})^{\epsilon/\alpha}}\right)^{\alpha} + (1-h)\left(\frac{e^{xt}R_g}{(Ae^{-xt})^{\epsilon/\beta}}\right)^{\beta} + d\left(\frac{R_b^*}{(Ae^{-xt})^{\epsilon/\gamma}}\right)^{\gamma} + f\left(\frac{R_g^*}{(Ae^{-xt})^{\epsilon/\mu}}\right)^{\mu}\right]^{1/\epsilon} C - (\delta_g + g - x)y_g$$

Next, we divide numerator and denominator in each fraction by A or A^* and extend the fractions by exponential terms for the foreign country:

$$\dot{y}_b = s_b\left[a\left(\frac{A_{-1}/A}{(Ae^{-bt})^{\epsilon/\varphi}/A}\right)^{\varphi} + h\left(\frac{e^{bt}R_b/A}{(e^{-bt}A)^{\epsilon/\alpha}/A}\right)^{\alpha} + (1-h)\left(\frac{e^{xt}R_g/A}{(Ae^{-bt})^{\epsilon/\beta}/A}\right)^{\beta} + d\left(\frac{e^{b^*t}R_b^*/A^*}{e^{b^*t}(Ae^{-bt})^{\epsilon/\gamma}/A^*}\right)^{\gamma} + f\left(\frac{e^{x^*t}R_g^*/A^*}{e^{x^*t}(Ae^{-bt})^{\epsilon/\mu}/A^*}\right)^{\mu}\right]^{1/\epsilon} C - (\delta_b + g - b)y_b$$

$$\dot{y}_g = s_g\left[a\left(\frac{A_{-1}/A}{(Ae^{-xt})^{\epsilon/\varphi}/A}\right)^{\varphi} + h\left(\frac{e^{bt}R_b/A}{(e^{-xt}A)^{\epsilon/\alpha}/A}\right)^{\alpha} + c\left(\frac{e^{xt}R_g/A}{(e^{-xt}A)^{\epsilon/\beta}/A}\right)^{\beta} + d\left(\frac{e^{b^*t}R_b^*/A^*}{e^{b^*t}(Ae^{-xt})^{\epsilon/\gamma}/A^*}\right)^{\gamma} + f\left(\frac{e^{x^*t}R_g^*/A^*}{e^{x^*t}(Ae^{-xt})^{\epsilon/\mu}/A^*}\right)^{\mu}\right]^{1/\epsilon} C - (\delta_g + g - x)y_g$$

Existence of a steady state requires that the fractions in these equations are constant. In a steady state we have $A/A(-1) = 1 + g$. The numerators for the home countries are constant according to equations (10a) and (10b). We use properties (i)-(iv) as follows. For the first denominator of the home country, with $b = x$ from (10a) and (10b), $\frac{\left(\frac{A(0)e^{gt}}{e^{bt}}\right)^{\frac{\epsilon}{\varphi}}}{[A(0)e^{gt}]} = e^{(g-b)t(\epsilon/\varphi)-gt}$. For $g = \frac{b}{\epsilon-\alpha}$ found above, we have

$(g-b)(\epsilon/\varphi) - g = \left(\frac{b}{\epsilon-\alpha} - b\right)\left(\frac{\epsilon}{\varphi}\right) - \frac{b}{\epsilon-\alpha} = \frac{b}{\epsilon-\alpha}\left(\frac{\epsilon}{\varphi}\right) - \frac{\epsilon b}{\varphi} - \frac{b}{\epsilon-\alpha} = \frac{b\epsilon}{(\epsilon-\alpha)\varphi} - \frac{(\epsilon-\alpha)\epsilon b}{(\epsilon-\alpha)\varphi} - \frac{b\varphi}{(\epsilon-\alpha)\varphi} = b\epsilon\frac{1-\epsilon+\alpha-\varphi/\epsilon}{(\epsilon-\alpha)\varphi}$. The first denominator term can only be constant if either b = 0, or ϵ = 0, or $1 - \epsilon + \alpha - \varphi/\epsilon$ =0. For the second denominator we get $b\epsilon\frac{1-\epsilon+\alpha-\alpha/\epsilon}{(\epsilon-\alpha)\alpha}$, simply from replacing ϕ by α. It is zero if either b = 0, or ϵ = 0, or $1 - \epsilon + \alpha - \alpha/\epsilon$ = 0. The third denominator term is constant if either b =0, or ϵ = 0, or $1 - \epsilon + \alpha - \beta/\epsilon$ = 0. If the third equation would be zero in all three cases, we would also have α = β = ϕ. This would lead to $\epsilon(1-\epsilon)+\epsilon\alpha-\alpha=(\epsilon-\alpha)(1-\epsilon)$= 0. This leads to either

α = β = ϕ with ϵ = 1, or α = β = ϕ = ϵ, the CES case for domestic variables. Replacing *b* by *x* would lead to the same results from the second equation.

For the foreign variables, we need a constant $e^{b^*t}(Ae^{-bt})^{\epsilon/\gamma}/A^*$, and the same for the last term with instead b* = x* and γ = μ. In terms of growth rates, we need $b^* - g^* + (g-b)\epsilon/\gamma = 0$ *or* $(g-b)\epsilon = \gamma(g^*-b^*)$. Using (10a), we get $(\frac{b}{\epsilon-\alpha} - b)\epsilon = \gamma(\frac{b^*}{\epsilon^*-\alpha^*} - b^*)$ *or*

$$b\frac{1-\epsilon+\alpha}{\epsilon-\alpha}\epsilon = \gamma(\frac{1-\epsilon^*+\alpha^*}{\epsilon^*-\alpha^*})b^*.$$

Again, this is a very special condition that would lead to constant denominators. For the domestic dynamic equations this can be summarized as

either (i) b = 0, or (ii) ϵ = 0, or (iii) α = β = ϕ with ϵ = 1, or (iv) α = β = ϕ = ϵ, in all four cases together with $b\frac{1-\epsilon+\alpha}{\epsilon-\alpha}\epsilon = \gamma(\frac{1-\epsilon^*+\alpha^*}{\epsilon^*-\alpha^*})b^*$.

From the dynamic equations of the foreign countries, we would get by symmetry the conditions

either (i) b* = 0, or (ii) ϵ* = 0, or, or (iii) ϵ* = 1 with α* = β* = ϕ*, or (iv) α* = β* = ϕ* = ϵ* in each case together with $b\frac{1-\epsilon+\alpha}{\epsilon-\alpha}\gamma^* = \epsilon^*(\frac{1-\epsilon^*+\alpha^*}{\epsilon^*-\alpha^*})b^*$.

Appendix VECMs: Backing out VES parameters

We assume *b* = *x*. This could allow backing out parameters values by combining the models formulas (9a,b,c) with the following empirical long-term relation from the VECM estimates of Soete et al. (2020b). In order to make the comparison comfortable we restate equations (9a-c):

$$logA = \frac{(1-\alpha)}{(1-\epsilon)}logR_b - \frac{\alpha b}{(1-\epsilon)}t + \frac{1}{(1-\epsilon)}\log\left(\frac{\epsilon F_2}{h\alpha C^{\epsilon}}\right) \quad (9a)$$

$$logA = \frac{(1-\beta)}{(1-\epsilon)}logR_g - \frac{\beta x}{(1-\epsilon)}t + \frac{1}{(1-\epsilon)}\log\left(\frac{\epsilon F_3}{(1-h)\beta C^{\epsilon}}\right) \quad (9b)$$

$$logR_b = \frac{(1-\beta)}{(1-\alpha)}logR_g + \frac{(\alpha b-\beta x)}{(1-\alpha)}t + \frac{1}{(1-\alpha)}log\frac{F_3}{F_2} + \frac{1}{(1-\alpha)}log\frac{h\alpha}{(1-h)\beta} \quad (9c)$$

For Austria, the long-term relations are two of the long-term relations for the domestic variables

logTFP = 1.338654logBERDST - 0.064463t - 10.39504

[57.4] [-15.7]

logBERDST = 0.215071logPUBST + 0.046615t + 5.841378

[18.8] [18.9]

Equations (9a) and (9c) allow us to find the values for the parameters by equating the coefficient formulas of these equations to the estimated coefficients. The system of equations providing the solution - for identical rates of depreciation as assumed in the process of making the data-, is

$\frac{(1-\alpha)}{(1-\epsilon)} = 1.338654; -\frac{\alpha b}{(1-\epsilon)} = -0.064463; \frac{(\beta-1)}{(\alpha-1)} = 0.215071; \frac{(\alpha-\beta)b}{(1-\alpha)} = 0.046615;$

$\frac{1}{1-(-2.9496)} log \frac{-2.9496\theta}{h(-4.2871)} = -10.39504; \frac{1}{1-(-4.2871)} log \frac{h(-4.2871)}{(1-h)(-0.13711)} = 5.841378.$

These are six equations from which we solve for α, β, ϵ, b, h and θ. The first four equations solve for the first four variables, which can be used in the last two equations. The Solution is:

$b = -5.9388\times10^{-2}$, $\alpha = -4.2871$, $\beta = -0.13711$, $\epsilon = -2.9496$, $h = 0.99999999999879$,

$\theta = 2.147\times10^{-18}$.

For Canada, the long-term relations are

logTFP = 0.265537logBERDST - 0.007909t - 2.606370

[12.19] [- 6.26]

logPUBST = 0.181181logGDP + 0.032668t + 7.045523

[13.8] [29.6]

logGDP = 0.542655logTFP + 0.021259t + 12.99533

[5.75] [43.34]

The last two equations yield

logTFP = 10.171logPUBST-0.37144t - 95. 608

The two equations for *logTFP* can be used to solve for the parameter values in (9a) and (9b). The equations to do so are as follows.

$\frac{(1-\alpha)}{(1-\epsilon)} = 0.265537; -\frac{\alpha b}{(1-\epsilon)} = -0.007909; \frac{(1-\beta)}{(1-\epsilon)} = 10.171; -\frac{\beta b}{(1-\epsilon)} = -0.37144;$

$\frac{1}{1-\varepsilon} log \frac{\epsilon\theta}{h\alpha} = -2.606370; \frac{1}{1-\varepsilon} log \frac{\epsilon\theta}{(1-h)\beta} = -95.608$. The solution is:

$b = -0.0367$, $\epsilon = -18.987$, $\alpha = -4.3072$, $\beta = -202.28$, $h = 2.4828\times10^{-806}$, $\theta = 1.3389\times10^{-829}$.

For France, the long-term relations are

logTFP = 0.246163logBERDST - 0.002546t - 2.879682

[-30.7] [6.3]

logBERDST = 1.053863logPUBST + 0.001897t - 0.299132

[-54.2] [-2.47]

Combining these coefficients with (9a) and (9c) yields four equations:

$\frac{(1-\alpha)}{(1-\epsilon)} = 0.246163; -\frac{\alpha b}{(1-\epsilon)} = -0.002546; \frac{(\beta-1)}{(\alpha-1)} = 1.053863; \frac{(\alpha-\beta)b}{(1-\alpha)} = 0.001897;$

$\frac{-2.14402\theta}{h0.22701} = e^{\{-2.879682\times(1+2.1402)\}}; \frac{h0.22701}{((1-h)0.18537} = e^{\{-0.299132\times(1-0.22701)\}}$. The solution is:

b = 3.5219×10^{-2}, ϵ = -2.1402, α = 0.22701, β = 0.18537; h = 0.3932, θ = -4.9315×10^{-6}.

For Italy, the long-term relations are

logTFP = 0.490138logBERDST - 0.018120t - 4.716875

[20.3] [- 9.7]

logBERDST = 1.215356logPUBST - 0.008533t - 1.809969

[19.9] [-2.84]

Combining these equations with (9a) and (9c) we get four equations again:

$\frac{(1-\alpha)}{(1-\epsilon)} = 0.490138; -\frac{\alpha b}{(1-\epsilon)} = -0.018120; \frac{(\beta-1)}{(\alpha-1)} = 1.215356; \frac{(\alpha-\beta)b}{(1-\alpha)} = -0.008533;$

$\frac{1}{1-\varepsilon} log\frac{\epsilon\theta}{h\alpha} = -4.716875; \frac{1}{1-\alpha} log\frac{h\alpha}{(1-h)\beta} = -1.809969$. The Solution is:

b = -3.9623×10^{-2}, ϵ = -29.464, α = -13.932, β = -17.147, h = 2.2529×10^{-12}, θ = 4.1837×10^{-75}.

For Portugal, the long-term relations are

logBERDST = - 0.235448logTFP + 0.091836t + 4.707567

[- 5.55] [26.7]

logTFP = 0.179110logPUBST - 0.004195t - 1.492951

[5.82] [-1.96]

Public R&D increases TFP, and, as a consequence, private R&D is reduced in the long run relations. Rearranging the first equation yields

$$\text{logTFP} = \frac{1}{-0.235448}\text{logBERDST} - \frac{0.091836}{-0.235448}\text{t} - \frac{4.707567}{-0.235448}$$

$\frac{(1-\alpha)}{(1-\epsilon)} = \frac{1}{-0.235448}; -\frac{\alpha b}{(1-\epsilon)} = -\frac{0.091836}{-0.235448}; \frac{(1-\beta)}{(1-\epsilon)} = 0.179110; -\frac{\beta b}{(1-\epsilon)} = -0.004195;$

$\frac{-6.57510\theta}{h33.173} = e^{\{(1+6.5751)\times(-4.707567)/(-0.235448)\}}; \frac{-6.57510\theta}{(1-h)(-0.35678)} = e^{\{(1+6.5751)(-1.492951)\}}$. The Solution is: b = -8.9068×10^{-2}, $\epsilon = -6.5751$, $\alpha = 33.173$, $\beta = -0.35678$, h = -2.2032×10^{-73}, $\theta = 6.6520\times10^{-7}$.

For all countries, the first four equations for the slopes are solved for the first four parameters and the last two equations for the intercepts then are solved for the last two parameters.

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