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**The serendipity theorem for an endogenous open economy
growth model**

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The serendipity theorem for an endogenous open economy growth model

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Abstract A Samuelsonian serendipity theorem for an endogenous growth model is derived. The formula for optimal population growth rate deviates from those of the model with exogenous population growth rates in a third best endogenous growth model of the Lucas type with imperfect international capital movements and human capital externalities. Calibration shows that the effect of variation of the exogenous population growth rates on other variables and the deviation of population growth rates from its optimal value are small. The reason is that labour supply, interest rates and technical change are endogenous. There is not much of an incentive for population growth policy unless Frisch parameters change with ageing.

Keywords: Open economy, endogenous growth, human capital, serendipity theorem, ageing. JEL codes: F43; J11,22,24;O11,15,41 .

Introduction

[1] re-considered the golden rule of [2]'s two-period version of the neoclassical growth model asking which of the values of population growth rates yields the highest welfare and found a first-order condition which he called the serendipity theorem. [3] showed that second-order conditions do not hold when Cobb-Douglas functions are used. [4] showed that second-order conditions are fulfilled if CES functions for utility and production have elasticity of substitution smaller than unity or else additional conditions may become necessary. [1]'s result was generalised by [5], [6], [7], and [8] in regard to endogenous fertility, exogenous and constant government debt, risky life time and medical expenditure.¹ We extend the theorem to endogenous growth with endogenous labour supply, imperfect international capital movements and human capital externalities in a third-best extension of [10] model with optimisation over all future generations by [11].

The Model

The maximisation programme for the consumers is

$$\begin{aligned} \max_{c_t, e_t, L_t, B_{t+1}, K_{t+1}, h_{t+1}} \sum_{t=0}^{\infty} \beta^t & \left(N_t \left(\frac{c_t^{1-\sigma}}{(1-\sigma)} - \xi \frac{\left(\frac{L_t}{N_t}\right)^{1+\vartheta}}{1+\vartheta} \right) \right. \\ & - \mu_t \left[N_t c_t + K_{t+1} - (1 - \delta_k) K_t - \omega_t (1 - e_t) h_t L_t - r_{kt} K_t - B_{t+1} + \left(1 + r \left(\frac{B_t}{Y_t} \right) \right) B_t \right] \\ & \left. - \mu_{ht} \left[h_{t+1} - F e_t^\gamma h_t - (1 - \delta_h) h_t \right] \right) \end{aligned}$$

¹ An interesting but only slightly related class of problems with endogenous population growth rates is discussed in [9]. The major differences are that in their paper population growth rates appear in the utility function, technical change is exogenous and a closed economy is considered. As our lists of literature are not overlapping we do not try to compare results.

The expression above shows the utility function of the entire population, N_t , where $0 < \beta < 1$ is the subjective discount factor, $\sigma > 0$ is the intertemporal elasticity of substitution for per-capita consumption, c_t , ϑ is the Frisch parameter for the active part of the population or labour supply, L_t , expressed as a share of the population, $\xi > 0$ is a parameter which measures the disutility of participation in the active population relative to the consumption part of utility. The households decide between spending their time in production ($1 - e_t$) for immediate output generation and education, e_t , to increase their productivity for later production. Income from labour with human capital h_t is $\omega_t(1 - e_t)h_tL_t$, the income from capital rent is $r_{kt}K_t$, and the debt from outside the economy's borders minus the interest and re-payment is, $B_{t+1} - \left(1 + r\left(\frac{B_t}{Y_t}\right)\right)B_t$. $r\left(\frac{B_t}{Y_t}\right)$ is the debt dependent interest rate. Spending on consumption is $N_t c_t$, and gross capital investment is $K_{t+1} - (1 - \delta_k)K_t$.

Consumption is not differentiated in regard to age or (not) working. This is implicit in the assumption of equal consumption of all for a given point in time. By implication, we do not model pay-as-you-go pension systems with defined benefits or contributions (see [12]) but instead forward looking savings and equal consumption, where the latter is a social equilibrium assumption similar to that in [13].

The economy is assumed to consist of output-producing firms and labour- and capital-supplying consumers. Output is formed by a Cobb-Douglas production function and is determined by physical capital, K_t , and efficient labour, $(1 - e_t)h_tL_t$. A human capital externality is added as, \bar{h}_t^ϵ , modelled after [10], to include the influence of the average skill level on the economy.

This forms the production function

$$Y_t = A(K_t)^{1-\alpha}((1 - e_t)h_tL_t)^\alpha \bar{h}_t^\epsilon \quad (1)$$

The demand for physical and human capital is determined in a firm which maximises profits:

$$\max_{(1-e_t), K_t} \pi = A(K_t)^{1-\alpha}((1 - e_t)h_tL_t)^\alpha \bar{h}_t^\epsilon - \omega_t(1 - e_t)h_tL_t - r_{kt}K_t$$

$$\omega_t = \frac{\alpha Y_t}{(1 - e_t)h_tL_t} \quad (2)$$

$$r_{kt} = (1 - \alpha) \frac{Y_t}{K_t} \quad (3)$$

Equations (2) and (3) represent first-order conditions, equating marginal productivity of labour and capital to wages for efficient labour and rental rates.

In the utility maximisation, given the initial value N_t , only the level of the population, N_{t+i} , appears.² It has three effects. Higher N_{t+1} leads to higher temporary utility $N_{t+1}u_{t+1}$; any given labour time L_{t+1} is shared among more people; more people have to be fed with additional consumption $N_{t+1}c_{t+1}$.³ The first-order condition for N_{t+1} , given N_t , and using (6) is

$$u_{t+1} \equiv \frac{c_{t+1}^{1-\sigma}}{1-\sigma} - \xi \frac{\left(\frac{L_{t+1}}{N_{t+1}}\right)^{1+\vartheta}}{1+\vartheta} \geq (\leq) - \xi L_{t+1}^{1+\vartheta} N_{t+1}^{-1-\vartheta} + c_{t+1}^{1-\sigma} \quad (4)$$

This condition is similar to that of [7], equation (6)), to which [8] has added the impact of medical expenditures and we add the labour supply term on the right-hand side. Collecting terms and dropping time indices $t+1$ yields

² Equations (1) - (3) do not have N_{t+1} terms.

³ The first and the last effect are very similar to that of an increased survival probability in [8].

$$c^{1-\sigma} \left(\frac{1}{1-\sigma} - 1 \right) (>) = (<) \left(\frac{L}{N} \right)^{1+\vartheta} \left(\frac{\xi}{1+\vartheta} - \xi \right) \quad (4')$$

(4') is a condition for the optimal population level. If $\sigma > (<) 1$, $\frac{\xi}{1+\vartheta} - \xi = \xi \frac{-\vartheta}{1+\vartheta} < (>) 0$ is required and fulfilled for any positive Frisch parameter.

As the Hamiltonian of the households dynamic problems defined above is already maximised for given N_{t+i} , when we derive with respect to N_{t+1} – analogous to indirect utility in [4] –, the envelop theorem applies and we only have to take the direct derivative with respect to N -terms and the second derivative with respect to N should be negative. For this we need $\xi - (1 + \vartheta)\xi < 0$ and because of (4') also $\sigma > 1$. For our iso-elastic utility case, production has unit substitution elasticities and labour supply should have a positive elasticity for $\xi > 0$.

Equation (4') will only be fulfilled for specific values of N_{t+1} as c and L are already optimally chosen by the household; it ensures that the two parts of utility are well balanced and consumption or labour supply are never too high or too low. However, N can adjust only slowly without migration and therefore temporarily this may perhaps hold only with inequality.

First-order conditions from household's utility maximisation for consumption c and labour L can be derived to find, together with (2),

$$c_t = \xi^{-1/\sigma} \left(\frac{L_t}{N_t} \right)^{-\vartheta/\sigma} \left(\alpha \frac{Y_t}{L_t} \right)^{1/\sigma} \quad (5)$$

This can be compared to our serendipity condition (4') in order to find the difference with the market equilibrium after making exponents comparable to (4'):

$$c_t^{1-\sigma} = \xi^{(\sigma-1)/\sigma} \left(\frac{L_t}{N_t} \right)^{-\vartheta(1-\sigma)/\sigma} \left(\alpha \frac{N Y}{L N} \right)^{(1-\sigma)/\sigma} = \xi^{(\sigma-1)/\sigma} \left(\frac{L_t}{N_t} \right)^{(-\vartheta-1)(1-\sigma)/\sigma} \left(\alpha \frac{Y}{N} \right)^{(1-\sigma)/\sigma} \quad (5')$$

The equality form of the serendipity condition (4') is

$$c^{1-\sigma} = \left(\frac{L}{N} \right)^{1+\vartheta} \xi \left(\frac{-\vartheta}{1+\vartheta} \right) \left(\frac{1-\sigma}{\sigma} \right) \quad (4'')$$

Equating right-hand sides of (4'') and (5') we get

$$\left(\frac{L}{N} \right)^{\frac{(\vartheta+1)}{\sigma}} \xi \left(\frac{-\vartheta}{1+\vartheta} \right) \left(\frac{1-\sigma}{\sigma} \right) = \xi^{(\sigma-1)/\sigma} \left(\alpha \frac{Y}{N} \right)^{(1-\sigma)/\sigma} \quad (6)$$

Other first-order conditions for the dynamic problem of the household are shown in [11]. Together with equations (1)-(3) they form a system of 11 equations for eleven variables. Successive insertion leads to a reduced system for the values for education time e and the debt-output ratio b depending on the population growth rate. These then determine the consumption share $X (= Nc/Y)$. All the values solving the model could be found without an assumption regarding the level of N as is well known for Lucas type of models. Equation (4) has no impact on the solution of the model for those variables. An increase of N will shift L , Y , K , B and C equi-proportionately. The model solution that compares to (6) in terms of growth rates is

$$\frac{(1+g_Y)}{(1+g_N)} = (1 + g_h)^{1+\frac{\epsilon}{\alpha}} \frac{(1+g_L)}{(1+g_N)} \quad (7)$$

where g_h is the endogenous growth rate of productivity h and it is constant in the steady state, which has no transition as the model can jump to its solution for e , b , $r(b)$, X , and all other variables. Whereas the model solution (7) links the growth rate ratios of Y/N and L/N linearly, in (6) comparable growth rates have exponents.

The growth rate of N can contribute to welfare by ensuring that the level and growth rates of c and L/N do not get too far apart. Equality of growth rates of both sides of (4') requires

$$(1 + g_c)^{1-\sigma} = \left(\frac{1+g_L}{1+g_N}\right)^{1+\vartheta} \quad \text{or} \quad 1 + g_N = \frac{1+g_L}{(1+g_c)^{\frac{1-\sigma}{1+\vartheta}}} \quad (4'')$$

The comparable equation from the model solution is again linear in growth rates of c and L/N .

$$(1 + g_c) = \left(\frac{1+g_L}{1+g_N}\right) (1 + g_h)^{1+\frac{\epsilon}{\alpha}} \quad (8)$$

Comparison of (6) and (7) as well as (4'') and (8) show that the serendipity condition is not redundant and could determine the population growth rate.

Calibration and population policy

The relevance of the serendipity theorem stems from the current problem of ageing based on the fall of population growth in the second half of the 1960s. The question then is how far population levels and growth rates are away from the optimum. We extend the calibration in [11] to calculate the optimal population growth rate from equation (4''). The calibration tries to get close to the average growth values for 14 OECD countries.

The parameters for production function (1) and those in the household utility maximisation are $\alpha = 0.6$, $\delta_h = 0.03$, $g_N = 0.002$, $F = 0.055$, $\gamma = 0.268$, $\epsilon = 0.834$, $\sigma = 1.06$, $\beta = 0.982$, $\vartheta = 3$. The debt elasticity of the

interest rate is a highly non-linear estimated function which depends on the debt/ratio itself (see [14]). Its values as well as those of the interest rate are part of the solution of the model. As indicated above the model can be reduced to a system in e and b depending on exogenous population growth rates. Table 1 shows results for alternative low population growth rates.

Table 1

g_N	g_L	$g(1+D)$	e_t	b_t	X_t	$r_t(1+\eta)1+g_c$	$1+gh$	$(1+g_v)/(1+g_N)$	$1+gN$ opt	
0.003	0.00256	0.00044	0.3953	0.0473	0.688	0.0507	1.0298	1.0129	1.0306	1.003002
0.002	0.00157	0.00043	0.3801	0.0383	0.694	0.0496	1.0287	1.0124	1.0295	1.001995
0.001	0.00059	0.00041	0.3658	0.0297	0.700	0.0484	1.0277	1.0120	1.0285	1.001000
0	-0.0004	0.00040	0.3523	0.0216	0.706	0.0473	1.0267	1.0116	1.0275	0.999995
-0.001	-0.00138	0.00038	0.3396	0.0137	0.713	0.0463	1.0257	1.0112	1.0265	0.999000
-0.002	-0.00237	0.00037	0.3276	0.0062	0.720	0.0453	1.0248	1.0108	1.0256	0.997997
-0.003	-0.00327	0.00027	0.3213	-2E-05	0.724	0.0444	1.0240	1.0106	1.0252	0.997085

The last column of Table 1 shows that the population growth rate according to the serendipity condition in terms of growth rates are hard to distinguish from those put in by assumption in the first column. At these low levels of population growth there is essentially no incentive for population policies.

Table 2 Model solutions for alternative Frisch parameters

ϑ	g_N	e	b	r	η	$1+r(1+\eta)$	$1+gh$	$1+g_L$
1	0.002	0.3838	0.0423	0.0473	0.0586	1.0501	1.012551	1.001122
2	0.002	0.3836	0.0447	0.0474	0.0615	1.0504	1.012545	1.001410
3	0.002	0.3837	0.0459	0.0475	0.0630	1.0505	1.012547	1.001559
4	0.002	0.3834	0.0466	0.0476	0.0639	1.0506	1.012539	1.001644
5	0.002	0.3834	0.0471	0.0476	0.0644	1.0507	1.012538	1.001703
6	0.002	0.3834	0.0475	0.0476	0.0649	1.0507	1.012537	1.001745
7	0.002	0.3833	0.0477	0.0476	0.0652	1.0507	1.012536	1.001777
8	0.002	0.3833	0.0479	0.0477	0.0654	1.0508	1.012536	1.001801

ϑ	g_N	$1+g_Y$	X	$1+g_c$	$1+g_w$	$1+g(1+D)$	$1+g_Y/1+g_N$	$1+gN$ opt
1	0.002	1.03141	0.69250	1.02937	1.017488	1.00088	1.02936	1.0015567
2	0.002	1.03170	0.69221	1.02965	1.017480	1.00059	1.02964	1.0018493
3	0.002	1.03185	0.69201	1.02980	1.017483	1.00044	1.02979	1.002000
4	0.002	1.03192	0.69198	1.02988	1.017472	1.00036	1.02986	1.0020862
5	0.002	1.03198	0.69193	1.02993	1.017470	1.00030	1.02992	1.0021459
6	0.002	1.03202	0.69189	1.02997	1.017469	1.00025	1.02996	1.0021887
7	0.002	1.03205	0.69185	1.03000	1.017468	1.00022	1.02999	1.0022209
8	0.002	1.03208	0.69183	1.03003	1.017467	1.00020	1.03002	1.002246

If the population gets older, this can perhaps be captured by a higher Frisch parameter if labour supply reacts more sluggishly to wage increases. Table 2 presents solutions of the model and for optimal population growth in the last column. Higher Frisch parameters imply higher optimal population growth rates between 0.0015 and 0.00225 when the model assumes a population growth rate of 0.002. Larger Frisch parameters therefore can be seen as an incentive for policies towards higher population growth.

Table 3 Model solutions for alternative rates of human capital depreciation

depr	gN	e	b	r	η	$1+r(1+\eta)$	$1+gh$	$1+gL$
0.01	0.002	0.341	0.540	0.074	0.347	1.09953	1.031	1.00091
0.02	0.002	0.361	0.271	0.061	0.235	1.07482	1.022	1.00123
0.03	0.002	0.384	0.046	0.048	0.063	1.05052	1.013	1.00156
0.033	0.002	0.391	-0.008	0.044	-0.013	1.04331	1.010	1.00166
0.034	0.002	0.393	-0.025	0.043	-0.042	1.04091	1.009	1.00169
0.035	0.002	0.396	-0.041	0.042	-0.072	1.03852	1.008	1.00172
0.036	0.002	0.398	-0.057	0.040	-0.105	1.03613	1.007	1.00175
0.037	0.002	0.401	-0.072	0.039	-0.140	1.03375	1.006	1.00179
0.038	0.002	0.403	-0.086	0.038	-0.177	1.03137	1.005	1.00182
0.04	0.002	0.408	-0.114	0.036	-0.259	1.02662	1.003	1.00189
0.05	0.002	0.435	-0.228	0.026	-0.878	1.00312	0.994	1.00221
0.06	0.002	0.464	-0.314	0.016	-2.216	0.98003	0.985	1.00254
0.07	0.002	0.495	-0.383	0.008	-6.371	0.95735	0.976	1.00288

depr	gN	$1+gY$	X	$1+gc$	$1+gw$	$1+g(1+D)$	$(1+gY)/(1+gN)$	$1+gN$ opt
0.01	0.002	1.077	0.683	1.075	1.044	1.00109	1.075	1.0019999
0.02	0.002	1.054	0.685	1.052	1.031	1.00076	1.052	1.0020002
0.03	0.002	1.032	0.692	1.030	1.017	1.00044	1.030	1.0019999
0.033	0.002	1.025	0.695	1.023	1.014	1.00034	1.023	1.0019999
0.034	0.002	1.023	0.696	1.021	1.012	1.00031	1.021	1.0020001
0.035	0.002	1.021	0.698	1.019	1.011	1.00028	1.019	1.0020000
0.036	0.002	1.019	0.699	1.016	1.010	1.00025	1.016	1.0019999
0.037	0.002	1.016	0.700	1.014	1.008	1.00021	1.014	1.0020001
0.038	0.002	1.014	0.702	1.012	1.007	1.00018	1.012	1.0019999
0.04	0.002	1.010	0.705	1.008	1.005	1.00011	1.008	1.0020000
0.05	0.002	0.988	0.723	0.986	0.992	0.99979	0.986	1.0020001
0.06	0.002	0.966	0.752	0.964	0.979	0.99946	0.964	1.0020002
0.07	0.002	0.945	0.800	0.943	0.966	0.99913	0.943	1.0019997

The most serious part of the problem of ageing may be the increase in the loss of human capital. In Table 3 we provide alternative solution of the model for different rates of human capital depreciation. Again, the deviations of the optimal population growth rate from equation (4'') from unity in columns ' $1+gN$ opt' are hard to distinguish from rounding errors although the solutions of the growth rates of c and L depend on the whole

model, which is first solved for e and b , and from there goes into the interest and growth rates, and they all depend on the rate of human capital depreciation. The human capital loss of ageing does not increase the incentive for population growth policies.

The comparison of the theoretical model with the additional serendipity result (4), (4'), (4'') and columns '1+gN opt' of Tables 1-3 show that conditions (4) - (4'') are not redundant and the choice of the population growth rates might lead to a better choice of growth paths. The result is interesting because all parts of the traditional golden rule are endogenous and optimised: the interest rate, the rate of technical change and labour supply growth. However, the deviation of the assumed from the optimal population growth rates are substantial only for variations of the Frisch elasticity, that is if ageing changes labour supply elasticities.

Equation (4) and its variations therefore broaden the serendipity results discussed in the introduction to the area of endogenous growth.

Conclusion

The serendipity theorem, suggesting that some rates of population growth may be better than others, turns out to be relevant in the third-best endogenous growth model with imperfect international capital movements as foreign debt, interest rates, labour supply growth and consumption paths are chosen optimally in the third best sense. We have considered the Benthamite case where the utility function is multiplied by the population size N . Other variants of utility functions could be analysed in the same way.

References

- [1] Samuelson, P.A. (1975) The Optimum Growth Rate for Population. *International Economic Review* 16 (3), 531-538.
- [2] Diamond, P. (1965) National Debt in a Neoclassical Growth Model. *American Economic Review*, LV (December), 1126-1150.
- [3] Deardorff, A.V. (1976) The Growth Rate for Population: Comment. *International Economic Review* XVII (June), 510-515.
- [4] Michel Ph. and Pestieau, P. (1993) Population growth and optimality: When does serendipity hold?. *Journal of Population Economics* 6:353-362.

- [5] Abio G. (2003) Interiority of the optimal population growth rate with endogenous fertility. *Economic Bulletin* 10(4): 1–7.
- [6] Jaeger, K., and Kuhle, W. (2009) The optimum growth rate for population reconsidered. *Journal of Population Economics* 22(1):23–41.
- [7] De la Croix D., Pestieau P., and Ponthière, G. (2012) How powerful is demography? The serendipity theorem revisited. *Journal of Population Economics* 25:899–922.
- [8] Felder, S. (2016) How powerful is demography? The serendipity theorem revisited. Comment on De la Croix et al. (2012). *Journal of Population Economics* 29: 957–967.
- [9] Boucekkine, R., Martínez, B. and Ruiz-Tamarit, J.R. (2017) Optimal Population Growth as an Endogenous Discounting Problem: The Ramsey Case. HAL Id: halshs-01579155.
- [10] Lucas, R.E. Jr. (1988). On the mechanics of economic development. *Journal of monetary economics*, 22(1), 3-42.
- [11] Gaessler, A.E. von, and Zieseimer, T. (2017) Ageing, human capital and demographic dividends with endogenous growth, labour supply and foreign capital, UNU-MERIT WP 2017-043.
- [12] Dedry, A., Onder, H., and Pestieau, P. (2016) Aging, Social Security Design, and Capital Accumulation, *The Journal of the Economics of Ageing*, 145-155 .
- [13] Muysken, J, Zieseimer, THW. (2014). The effect of immigration on economic growth in an ageing economy. *Bulletin of Applied Economics*, 1(1), 37-65. http://www.riskmarket.co.uk/bae/journals-articles/issues/1_1_3/?download=attachment.pdf.
- [14] Gaessler, A.E. von, and Zieseimer, T. (2016) Optimal Education in Times of Ageing: The Dependency Ratio in the Uzawa-Lucas growth model, *The Journal of the Economics of Ageing* 7, 127-142. doi: <http://dx.doi.org/10.1016/j.jeoa.2016.03.001>.

Appendix (not for publication)

Derivation of (6):

$$\left(\frac{L}{N}\right)^{1+\vartheta} \xi \left(\frac{\vartheta}{1+\vartheta}\right) \left(\frac{1-\sigma}{\sigma}\right) = \xi^{(\sigma-1)/\sigma} \left(\frac{L_t}{N_t}\right)^{(-\vartheta-1)(1-\sigma)/\sigma} \left(\alpha \frac{Y}{N}\right)^{(1-\sigma)/\sigma}$$

$$\left(\frac{L}{N}\right)^{\frac{(\vartheta+1)(1-\sigma)+(\vartheta+1)\sigma}{\sigma}} \xi \left(\frac{\vartheta}{1+\vartheta}\right) \left(\frac{1-\sigma}{\sigma}\right) = \xi^{(\sigma-1)/\sigma} \left(\frac{L_t}{N_t}\right)^{(-\vartheta-1)(1-\sigma)/\sigma} \left(\alpha \frac{Y}{N}\right)^{(1-\sigma)/\sigma}$$

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