#2013-009

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Designing an Optimal 'Tech Fix' Path to Global Climate Stability:
R&D in a Multi-Phase Climate Policy Framework

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First draft: 25 August, 2012
This version: 31 January 2013

ABSTRACT

The research reported here gives priority to understanding the inter-temporal resource allocation requirements of a program of technological changes that could halt global warming by completing the transition to a "green" (zero net CO2- emission) production regime *within the possibly brief finite interval that remains before Earth's climate is driven beyond a catastrophic tipping point*. This paper formulates a multi-phase, just-in-time transition model incorporating carbon-based and carbon-free technical options requiring physical embodiment in durable production facilities, and having performance attributes that are amenable to enhancement by directed R&D expenditures. Transition paths that indicate the best ordering and durations of the phases in which intangible and tangible capital formation is taking place, and capital stocks of different types are being utilized in production, or scrapped when replaced types embodying socially more efficient technologies, are obtained from optimizing solutions for each of a trio of related models that couple the global macro-economy's dynamics with the dynamics of the climate system. They describe the flows of consumption, CO_2 emissions and the changing atmospheric concentration of green-house gas (which drives global warming), along with the investment dynamics required for the timely transformation of the production regime. These paths are found as the welfare-optimizing solutions of three different "stacked Hamiltonians", each corresponding to one of our trio of integrated endogenous growth models that have been calibrated comparably to emulate the basic global setting for the "transition planning" framework of dynamic integrated requirements analysis modeling (DIRAM). As the paper's introductory section explains, this framework is proposed in preference to the (IAM) approach that environmental and energy economists have made familiar in integrated assessment models of climate policies that would rely on fiscal and regulatory instruments -- but eschew any analysis of the essential technological transformations that would be required for those policies to have the intended effect. Simulation exercises with our models explore the optimized transition paths' sensitivity to parameter variations, including alternative exogenous specifications of the location of a pair of successive climate "tipping points": the first of these initiates higher expected rates of damage to productive capacity by extreme weather events driven by the rising temperature of the Earth's surface; whereas the second, far more serious "climate catastrophe" tipping point occurs at a still higher temperature (corresponding to a higher atmospheric concentration of CO2). In effect, that sets the point before which the transition to a carbon-free global production regime must have been completed in order to secure the possibility of future sustainable development and continued global economic growth.

JEL codes: Q540, Q550, O310, O320, O330, O410, O440.

Keywords: global warming, tipping point, catastrophic climate instability, extreme weather-related damages, R&D based technical change, embodied technical change, optimal sequencing, multi-phase optimal control, sustainable endogenous growth.

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1 Introduction: Climate Instability and Environmental Policy Research

Economic developments thus far in human history have been linked closely with progress in methods for the bulk conversion into useful work of the energy stored in carbon-based fuels.[1] Burning wood, coal, oil, and natural gas gives rise to CO2-emissions that, together with releases of other greenhouse gasses (GHG's) like methane, are now thought to be responsible for the considerable warming of the earth's atmosphere since the industrial revolution of the late eighteenth century and the worrying prospect of that upward trend continuing for years to come.[2]

1.1 Motivation: climate science and climate policy

The implied future consequences are "bad news" on a number of related counts: sea levels will rise, tropical diseases will become more wide-spread, storms will be more violent, patterns of rainfall will change (affecting agriculture), and fresh-water supply shortages will become a problem due to global glacier retreat, and so on.[3] Most of these consequent changes represent significant costs to society en route to the potential emergence of catastrophic climate instability.[4] But, more worrying still is the possibility that the environmental changes set in motion by the warming of Earth's land surface and oceans may impart a self-reinforcing momentum and thus accelerate the rise of global mean temperature to a pace faster than that resulting solely from anthropogenic releases of GHG.

The past 20 years have brought revolutionary advances in the study of climate history based on the deep-ice cores of the millennia between the last glacial maximum and the opening of the (present) Holocene era.[5] Among the many important findings are those that have given greater plausibility and disturbing palpability to the conjectured existence of global climate "tipping points" that can trigger "abrupt" changes in the climate system.[6] To allow the Earth's

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[1] That is to say, "work" in the sense that the term is used in physics. See Cipolla (1972) for an elegantly concise and penetrating presentation of this theme in economic history.

[2] We put aside here discussion of remaining scientific uncertainties regarding the pace of warming and the extent to which it is attributable to anthropogenic sources, and observations on the role of active climate change denial (whether from ignorance, innate skepticism or strategic perception of material self-interest). See the comments and references in David and van Zon (2012: esp. sect. 2).

[3] See, e.g. Dyer (2010) and Lynas (2007) on the disruptive social and political sequelae. Lynas provides a journalistic view on the consequences for the world of up to 6 degrees additional warming for each degree of extra warming, while Dyer provides scenarios regarding the struggle for resources necessary to fullfil basic needs, like water and arable (and habitable) land, as these are likely to arise with on-going global warming.

[4] See the Stern Review (Stern 2007), and ICPP (2007); also David and van Zon (2012: sect 2.,1) for further discussion of the current science consensus of the literature on global warming, ecological disruptions, including and human welfare damages – upon which the following paragraph draws.

[5] See e.g., Alley (2002), Burroughs (2005), Cronin (2009).

[6] The term "tipping point" has been used in discussions of global change to describe a variety of phenomena, including the appearance of positive feedbacks, reversible phase transitions, phase transitions with hysteresis effects, and bifurcations where the transition is smooth but the future path of the system depends on having been perturbed at a critical point. On abrupt climate changes and "earth tipping points" see Alley , Marotzke, Nordhaus, Overpeck et al.(2003); National Research Council (2003). Stern (2007: pp. 11-14) provides an overview of positive feedback processes involving reduction of albedo through reduced ice-coverage of the arctic regions, thawing of permafrost and induced release of methane. The broader term "tipping elements" is introduced by Lenton, Held, Kriegler et al. (2008) to describe subsystems of the Earth's geophysical system that are at least sub-continental in scale and under certain conditions can be switched into a qualitatively different state by small perturbations. The tipping point is a structural feature of such a sub-system, corresponding to the critical point in the forcing process at which the future state of the global climate system would be qualitatively altered – but not necessarily involving a self-reinforced, irreversible state change. The Atlantic Meridional overturning in the thermohaline circulation, the West Antarctic ice sheet, the Greenland ice sheet, and the Amazon rainforest and the El Nino/Southern Oscillation are the sub-systems identified

mean global temperature to rise above a critical threshold of that kind would launch a self-reinforcing and irreversible warming process that could manifest itself eventually in catastrophic climate "flickering" – characterized by recurring abrupt climate changes, switching back and forth at high frequency between spatially uncorrelated bouts of pronounced warming and cooling.[7]

More than simply adding another ground on which the "precautionary principle" calls urgently for mitigation of CO_2 emissions, that existential threat undermines sanguine presuppositions that "adaptations" by contemporary societies to a continuing gradual rise in global mean temperature would eventually bring humanity to a significantly warmer but nonetheless viable equilibrium environment; that that current climate policy should be aimed to effect a correspondingly gradual, economic welfare-maximizing approach to that distant but attainable goal. The alternative prospect now more clearly envisaged by many climate scientists –namely, that the rising trend of mean global temperature could take Earth's climate system beyond a "catastrophe tipping point" without the consequences manifesting themselves plainly for some time to come -- raises grave doubts about the formulaic economic advice that continue to be widely espoused. Climate policy, it is said, should devise instruments to guide the world economy along an optimal path that balances the present value of social welfare sacrificed by actions to restrict CO_2 emissions, against the present value of the future net social welfare gains resulting from slowing the accumulation of atmospheric CO_2 – just enough to allow modern civilization sufficient time to adapt itself to environments that will be on the whole rather warmer, but in which adaptation will bring benefits to set against what has been lost.[8]

There is much to be said, instead, for timely actions aimed to sharply slow and eventually halt global warming by reducing the flow of GHG emissions, as well as to increase the capacity of human societies to carry on in the (hopefully viable) circumstances that the peoples of the world will face when they succeed in stabilizing its many climates. "Defensive" adaptation to curtail the extent of the economic and social damages wrought by increasingly frequent and severe storms, coastal flooding and drought must, undoubtedly, be made part of the response to global warming -- if only because much of the warming that will drive them, and accompanying alterations of future regional weather patterns are "in the pipeline" already, as a consequence of past anthropogenic GHG emissions.[9] With the current level of atmospheric concentration of CO_2

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most frequently experts as harboring "large-scale discontinuities" – i.e., dangerous "tipping point" potentialities under anthropogenic global warming.

[7] See, Hall and Behl (2006) on the "Clathrate Gun" hypothesis that has been advanced in explanation of abrupt climate change and the onset of the phenomenon of "climate flickering" at the end of the last ice age, and further discussion with other references in David and van Zon (2012:sect. 2.1).

[8] See Hall and Behl (2006) on the persisting failure of the economic literature on integrated assessment models (IAMs) to address the implications of climate scientists' conclusions and explanatory conjectures based on the paleoclimate evidence of "climate flickering"; esp., pp. 461-462, for a detailed critique of the representation of the climate sub-system in Nordhaus and Boyer's (2000) updating of the original (Nordhaus 1994) DICE model. The assumption that radiative forcing due to the accumulation of atmospheric CO_2 would drive a smooth transition to a higher equilibrium temperature of the Earth's surface is retained in the latest update of DICE (see Nordhaus 2010), as well as in the annualized version of the model that Cai, Judd and Lontzek (2012) create en route to SDICE, their stochastic control version of DICE – which in other respects constitutes a significant advance in the IAMs literature.

[9] As will be seen (in sect. 3.2), the third of the models presented here allows for warming –driven damages to global productive capacity, but does not also consider the option of undertaking "defensive" capital formation that would reduce those damages. Such investment should be viewed as a form of "adaptation" and although its importance has become more widely appreciated, it remains the case that most of the available integrated policy assessment models do not explicitly treat that option. A noteworthy exception is "AD-DICE", an IAM that allows explicit quantitative examination of the trade-offs between adaptation and mitigation of CO2 emissions by means of carbon taxes. See de Bruin, Dellink and Tol (2009), and notes in sects. 3.2 and 4.2, below, for further discussion.

and the high rate of continuing emissions, it is becoming more and more doubtful that global warming can be kept from adding a gain of more than 2º Kelvin to the mean surface temperature that prevailed throughout the recorded pre-industrial millennium. Such a large temperature gain is considered to be a threshold that ought not be crossed, for fear of triggering runaway global warming and the attendant societal chaos and widespread human losses.[10]

Unfortunately, the mere acknowledgement of the existence of climate catastrophe tipping points is not enough to ensure that CO_2 and other GHG emissions will be sufficiently reduced. The BRIC group of industrially developing countries has maintained astonishingly rapid rates of economic growth during the past two decades, relying heavily on fossil fuel sources of energy. The peoples of these nations think it entirely appropriate that they should continue not only to lift their remaining masses from poverty, but advance toward levels of economic welfare comparable to those in the West.   It goes without saying that the nations of the West also feel similarly entitled, not only to maintain but to also pursue still higher welfare levels. As a consequence, the global provision of energy will be under pressure to accommodate these widely shared aspirations. This, unfortunately, is already reflected in the fact that many of the recently built electricity generating plants in China and India, and hundreds more proposed for construction there and elsewhere, are to be coal-fired.[11] Were that prospect not discouraging enough, the 2011 tsunami and Fukushima Daiichi nuclear disaster has led authorities in Japan, and the governments of other countries to begin taking steps to reduce their economies' dependence on nuclear power; the closing of a significant number of nuclear power stations has been announced in Germany, raising concerns that the electric generation capacity thus abandoned will be replaced with new coal-fired plants.[12] In short, there are strong political and economic pressures that continue to promote adherence to a highly carbon-intensive growth–path for the global economy--with mounting risks of triggering runaway climate change.[13]

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[10] See, e.g., Hansen, Sato, Ruedy et al. (2006: p.1): "Comparison of measured sea surface temperatures in the Western Pacific with paleoclimate data suggests that this critical ocean region, and probably the planet as a whole, is approximately as warm now as at the Holocene maximum and within ≈1°C of the maximum temperature of the past million years. We conclude that global warming of more than ≈1°C, relative to 2000, will constitute ''dangerous'' climate change as judged from likely effects on sea level and extermination of species."

[11] According to a World Resources Institute report (see Yang and Cui (2012)) a recent WRI survey found 1,199 coal-fired plants currently proposed globally in 59 countries, over two-thirds of them in China and India with aggregate capacity representing about three-quarters of the projected global additions to installed mega-wattage. The number of projects planned in India (455) exceeds those in China (363), but the latter's plants are on average larger, so that their total installed mega-wattage would be about 10 per cent greater than India's (510K MW). Whether these all these plans will materialize is another matter, however: in China stricter new regulations in response to air pollution problems, and prospective water shortages (for cooling) may militate against it, and the Indian power-plant boom of the past decade ended with many projects having been discontinued.

[12] In terms of the implications for global warming, coal is among the worst of available fossil fuel options: the ratio of grams of CO_2 per joule of energy obtained from coal is 1.69 times higher than that from natural gas (cf. *Wikipedia*: "Energy density"). That comparison, however, substantially understates the comparative "dirtiness" of coal-fired vis-à-vis natural gas electric power generation plants over their respective full life-cycles (from ground-breaking to fuel sources, waste management and return to greenfield site): three meta-analyses of many studies concur in placing the corresponding ratio of CO_2 g per KWh_e for coal-fired plants (with scrubbers) to that for natural gas-fired plants in the narrow range 1.96-2.17 (cf. *Wikipedia*:"Life-cycle greenhouse-gas emissions of energy sources." In addition to being climatically "dirty," and emitting soot and other health-injuring pollutants, the trouble with building new coal-fired power plants is that the low international price of coal is likely to be kept down by the expansion of still cheaper substitutes like natural gas, and increasing coal extraction rates (induced by expectations of future carbon taxes). This would tend keep these durable power station's marginal costs from rising as they age, so that they could continue yielding quasi-rents and remain in commercial operation for many decades, unless forcibly shut down by regulatory actions.

[13] Obviously, that threat may be mitigated by fundamental changes in lifestyles, at least for the well-to-do part of the global population that presently are major users of energy from fossil fuels. But it is hard to imagine that such an impulse would gather political force spontaneously from a transcendent ideological movement, or that it could be

With humanity today seemingly headed almost unavoidably towards a worrisomely problematic future climate, it is all the more pertinent to thoroughly explore what economic theory informed by climate science can tell us, *firstly* about the broad requirements for satisfying the desire to sustain development and the future growth of economic welfare by allocating resources so as to avert drifting irreversibly into a catastrophically transformed global environment; *secondly* what the role of both existing and new technologies and their embodiment in irreversible investment may be in effecting a transition from "business-as-usual to a viably low carbon production regime that would stabilize the climate; and *thirdly* whether such a "tech fix" program would be technically and economically feasible to execute within the finite time limits that may remain to design and implement the necessary steps in that program before arriving at a tipping point into irreversible "runaway" climate change (*TPIRCC*, hereinafter).

Only when we have a clearer vision of what would be required to travel the technological path to a stabilized global climate that would be least costly in terms of human welfare (i.e., "first-best" welfare-optimal), will we then be in a better position than now to realistically consider the policy measures that might be able to realize such a program; or, to stating the nature of latter challenge more precisely, to re-focus negotiations about national and international agreements on continuously monitor-able actions to implement a socially and politically feasible approximation to the idealized, first-best welfare optimum path toward sustainable development in stabilized global climate.[14]

Unfortunately, in our view, much too much time already has passed during which serious people approaching this dauntingly formidable policy-design challenge have tried to avoid tackling it head-on, and have opted instead for a different strategy --based on trying to deal with the challenge of global warming by directly addressing its root economic causes. These may be found in humans' historical propensity to burn carbon, and to continue doing so without regard for the cost in terms of environmental harms that such activities impose upon others, and will eventually impose far more heavily on still unborn generations. Such costs, falling upon others-- whether through immediate localized pollution of the nano-particle laden air inhaled by the population in urban areas, or the rising concentration of atmospheric GHG and all of its present and future global consequences–-are labeled as "externalities". They are not reflected fully in "free market prices" of carbon fuels, and hence do not register among the private costs of those engaged in combusting those materials.

Starting from this diagnosis of the source of the climate problem, a substantial consensus among environmental and energy economists has found the indicated course of climate policy response to be elegantly simple: directly address the "externality" of GHG emissions by using fiscal or a combination of regulatory and fiscal instruments to set taxes so as to generate market prices of carbon (or for licenses to burn fossil fuels) that will fully reflect the marginal social costs of using those materials as sources of energy.   Then, it is said, the workings of the free market will take care of the rest of the problem; raising the carbon price will create private incentives to reduce CO_2 emissions in ways that will be least-costly to the actors; "getting the prices right" now and in the future therefore will be sufficient to fix the "externality" that has allowed economically self-interested humans to set their planet on its warming course.

There have been real advances in analyzing the likely effects of using fiscal and regulatory instruments in the most welfare efficient ways to fix the CO_2 emissions "externality,"

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effective without the facilitation and reinforcement of extensive technological changes in the global production regime.

[14] See Schelling (2009) on the distinction between setting outcome targets and agreeing on scheduled actions.

indeed, advances upon which this paper is able to build. Yet, at the same time there has been a discernible and unwarranted reluctance among economists to examine the technological requirements of what "the market" would have to accomplish by altering private incentives to curtail the emissions of CO2. It is as if those researchers had said, perhaps not aloud: "I'm an economist not an engineer, and since I have good cause to expect that markets will do an efficient job of resource allocation once we have removed the troublesome externality, why do I have to be concerned about the details of the technological solutions that rational economic agents will arrive at when guided by the right prices?"

The evident problem with this line of response is that "getting the global price of carbon right," and keeping it right, are not so simple tasks for "institutional engineering" and international political economy as it is convenient to imagine while formulating and solving those integrated assessment models (IAMs).[15] Secondly, if economists' attention were to turn away from the problem of the CO2 emissions externality for a moment or two, they would remember that there are other, well-known externalities that are likely to thwart the spontaneous emergence of socially efficient decentralized responses to rightly-priced carbon.

Two such externalities are so familiar that they may be mentioned without further elaboration: existing technological knowledge must be implemented, and implementation under actual field conditions may require investment in researching incremental adaptations and interactions between providers of equipment embodying that knowledge and its users. But there are knowledge benefits from learning-by-doing, and investments in research that are difficult if not impossible for private agents in competitive markets to appropriate completely, giving rise to well-known information transaction externalities and "spill-overs" that lead to socially inefficient resource allocation.[16] Likewise, investment decisions relating to the implementation of existing technologies (and of recent innovations, *a fortiori*) may be distorted by the existence of other knowledge spill-overs from learning by-doing, and learning-by-using.[17]

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[15] Beyond the issues of negotiating and enforcing coordinated international commitments to a cap-and-trade mechanisms that already have been exposed by the experience with the UN Kyoto Protocols and the attempts to go beyond them at the 2009 Copenhagen Conference, there is the another specific problem the dependence of such mechanisms on markets for "licenses" or permits (either permits to emit CO2 and other GHGs in the case of so-called downstream cap-and trade), or licenses allowing first vendors to transact in combustible forms of carbon (in the case of upstream cap-and-trade). Transferable permits and licenses constitute a financial asset. They are desirable because it is envisaged that they would be traded in markets in which price adjustments would be endogenous and automatic (not requiring repeated government actions, assuming the caps were set correctly at the outset). Financial assets, however, will permit securitization, and bonds will engender the formation of markets for derivatives, and so on. It is not a trivial problem to arrange for the monitoring and regulating of the issue of these new financial instruments, and to integrate global markets for the varied types of licenses, so that the current and expected future prices of the underlying assets will be able to become wildly wrong. Indeed, the difficulties of achieving such a desirable result and the costs of failure to do so (again) have been demonstrated in the destructive financial crisis of 2008-2010 and its economically and socially painful aftermath. Aside from these practical institutional details, and the political problems of arranging for national legislative bodies constituted of political representatives with short career horizons to commit themselves and their successors to distant future schedules of taxes on fossil fuels, there are serious questions about the general equilibrium effects of a commitment to raise the future relative price of those energy sources. These include possible adverse effects on private incentives to invest in improving the efficiency of carbon-based technologies, and perverse "anti-conservation" responses from owners of fossil fuel deposits that could vitiate the direct effect of the carbon tax or might even force down the near term after tax price of using carbon energy sources. For references and more detailed discussion see David 2009, David and van Zon (2012: sect. 1), and van der Ploeg and Withagen (2010),specifically on potential anti-conservation ("Green Paradox") effects of carbon-pricing.

[16] For recognition of the existence of "the second externality" in the context of environmental policy, see the discussion of the so-called "appropriability problem" in Jaffe, Newell and Stavins (2003: section 3, esp. pp. 471-473).

[17] This has long been recognized in the literature on the economics of science and technology policy, and particularly in contributions that reflect "evolutionary" economic analysis and modeling (see, for discussion and references, e.g., David, 1992; Aghion, David and Foray, 2008). The corollary is that with more than one externality there will be a need for more than one public policy mechanism to correct likely misallocation of resources. The relevance of this in the

1.2 The Research approach and methods: An overview

There is a more fundamental reason for economists to devote more serious attention to examining technology policy options, and in this paper it proceeds simply from the fact that in one way or other technological implementations of policy measures designed to curtail warming will be required. Investment (*either in generating future knowledge or in tangible production facilities embodying mature and novel technologies*) is the '*conditio sine qua non'* for a successful transition towards a sustainable future. The transition towards sustainability will therefore be determined by finding the right balance between two important aspects of investment: on the one hand we have to face the fact that the irreversibility of investment implies a certain degree of inertia to change, while on the other hand investment is literally the carrier of technological progress and so 'enables' (productivity-) change.

This "double-role" of investment underlines the importance of the timing of investment decisions: it is unwise to invest too early because one runs the risk of missing out on potential productivity improvements still to come, and neither should one invest too late because of the rising opportunity cost of continuing to use old technologies instead of new, superior, ones. This setting naturally gives rise to such questions as how long to continue using and investing in present carbon-based technologies, how much and how long to spend efforts on improving 'new' carbon-free technology, and when to stop improving such carbon-free technologies and start implementing and using them, this all in the face of having to stop cumulative emissions just in time and just below the *TPIRCC*. In other words, the design of emission-mitigating policies calls for "thinking-in-time" about complicated questions about which investment options to consider, how much investment has to be undertaken, and in what order it will be best to do it. It calls for beginning a stage before assessing the effects of alternative policy choices, the stage that deals with questions that engineers and planners of systems with improved performance property refer to as "requirements analysis".

It also will be instantly recognized as "planning analysis" posited on the assumed existence of a socially benevolent and omniscient planning agent, and therefore setting aside for the purposes of the analysis all considerations of the problems of how to implement the actions required to achieved the system's desired performance if the system in question happens to be one in which resource allocation is decentralized and reflects the distributed decisions and behaviors of many human agents. Optimal planning models, whether of the deterministic or the stochastic variety, are familiar tools in modern economics and our research simply (or not so simply) extends their use in the literature of macroeconomic growth models by applying it to examine a global economy whose planner has to answer to resource allocation questions that are complicated by having to take into account the dynamic interconnections between his economy and the geophysical system in which it happens to be inextricably embedded.

To find answers to these questions, we formulate an optimum control model that borrows heavily from the AK-model from the endogenous growth literature (cf. Rebelo (1991) in particular), but that also expands upon this AK-setting in a number of ways. First, we allow for different technologies that can be used either next to each other or sequentially. Secondly, a technology is characterized not only by its capital productivity, but also by CO_2-emissions per unit real output. In the initial business-as-usual phase, "carbon-free" technologies will be taken to allow production with zero *net* emissions of CO_2, although doing so at higher unit costs of

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context of climate policy seems only quite recently to have impressed itself upon leading contributors to the theoretical literature on endogenous economic growth (see Acemogulu et al., 2012).

capital (i. e., being characterized by a lower average and marginal productivity of capital).[18] The term "net emissions" here refers to the flow of production-generated CO2 that is in excess of the natural abatement capacities of the Earth's oceans and forests, and, rather than being trapped and stored there, eventually adds to the concentration of atmospheric GHG.

Third, we allow for the deactivation of existing technologies, i.e. the 'scrapping' of existing technologies, as in the vintage literature. We show how the time of deactivation of existing capacity depends on technological parameters but also on emission characteristics, in combination with the shadow price of emissions. The latter suggests that the position of the climate catastrophe "tipping point" directly influences the optimal timing of such moves towards completing the switch away from carbon-based production -- through its impact on the shadow price of emissions. An exogenous shock, shrinking the space left for further accumulation of atmospheric CO2 (such as would come in the form of compelling evidence that the onset of a runaway process of warming could be triggered by a rise in the CO2 –equivalent GHG concentration level beyond 430-450 ppmv, now that we are already close to the 400 ppmv) would send the shadow price of emissions suddenly upwards, forcing drastic actions to curtail the output of consumption goods toward subsistence-satisfying levels so that as much of the operating capacity that remained could be used to rapidly build up the stock of carbon-free capital.

Fourth, we allow for endogenous R&D based technical change. This requires a specification of the R&D function that is different from the ones found in Romer (1990), or Aghion and Howitt (1991), for example, because technical change in our model setting is not meant to compensate for the loss in capital productivity resulting from capital accumulation under neo-classical conditions, as in Lucas (1988) and Romer (1990), for instance. Rather, in an AK-setting, capital productivity is constant by assumption, and so technical change specified in the usual way would produce a continuously accelerating growth rate rather than just a higher, but still constant, rate of growth. We will come back to this in more detail later on. Fifth, we explicitly focus on the timing of the switches between investment in the one technology and in the other, and on the timing of the deactivation of old technologies, since the deactivation and activation of technologies that differ w.r.t. their emission-coefficients have a direct impact on the macro-emission rate and therefore on the time left until the *TPIRCC* will be hit.

As the foregoing overview of our approach implies, a central issue in the model to be presented here is the fact that the transition towards a carbon-free production system entails a switch in the deployment of production technologies that require the buildup of carbon-free capacity and the simultaneous rundown of carbon-based capacity, simply because the one type of capacity cannot be changed into the other type of capacity without new tangible capital formation. Therefore we have opted for an AK-model setting (cf. Rebelo, 1991) in which we allow for two technologies that both can produce output, one of which results in CO2 emissions whereas the other does not.

The present paper summarizes part of the work conducted by the authors on the construction (and further extension) of a multi-phase transition models incorporating the concepts of changes in technologies that are embodied in tangible and durable capital goods, and of the irreversibility of investment decisions. From these basic premises it follows that the "smooth"' transition toward a carbon-free future will need to be prepared by means of the

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[18] The latter is a necessary assumption to make the model consistent with the observation that the present state of the economy is characterized by the intensive use of carbon-based energy rather than carbon-free energy. If the capital productivity of the latter technology would exceed that of the former, then the carbon-free technology would be superior to the carbon-based technology on all accounts, which is not really the case.

accumulation and subsequent run down of carbon-based production capacity, simply because capital, whether carbon-based or carbon-free, is a produced means of production. The focus of the resulting models therefore is on the selective build-up and deactivation of different types of capital stocks, the time this takes, and the implications of this process for the development over time of welfare specified in terms of the flow of consumption.

The premise that embodiment in physical capital goods is necessary to implement changes in the technologies that would lower the global production regime's carbon-intensity, as well as to increase the capacity of the carbon-based capital stock, therefore is a distinctive feature of the modelling approach pursued here. Although energy technologists and engineers have in effect long recognized that the "embodiment" of techniques in fixed reproducible structures and equipment cannot be ignored when considering the impact of technical innovations in energy supply systems,[19] our explicit recognition of this in the very structure of the model presented here (and its account of the dynamics of the optimal climate-stabilizing transition path) represents a major departure from the ways in which the effects of endogenous technological change have been treated in previous economic contributions to the integrated assessment of climate policy measures.[20]

Our multi-phase modeling approach also offers some novel additions to the existing but still relatively small literature concerned with the optimal timing of switches among alternative production technologies. Several points of comparison with the present analysis are worth remarking upon in regard to three notable preceding contributions in this vein, by Tahvonen and Salo (2001), Valente (2009) and Schumacher (2011),[21] Tahvonen and Salo (2001) focus on the timing of the switch between alternative resource extraction technologies that differed in their variable costs, whereas Valente (2009) examines the switch between two macro-production technologies in a setting without irreversibility of investments in production capacity nor endogenous technical changes resulting from investments in R&D. The location of a single optimal switching moment in Valente's (2009) analysis is determined a standard dynamic optimization setting involving a CIES utility function.   By contrast, our supposition that formation of physical production capacity embodying each technology must have occurred

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[19] The development and use of energy technologies is viewed as an integrated system comprising research discoveries and inventions, the creation of commercial products and processes, their initial deployment and adoption into commercial operations, and subsequent wider diffusion – the view embraced recently by the Report of the President's Council of Advisors on Science and Technology (PCAST, 2012). Accordingly, Ernest Moniz (2012: p. 82), former Undersecretary of the U.S. Department of Energy and a PCAST member, emphasizes the importance of tangible fixed capital formation in considering policies designed to stimulate "energy technology innovation": "Adoption and diffusion are the stages at which materiality of [novel] products and processes are realized (or not). *Innovation, as I use it here, refers to the end-to-end system including market diffusion, not front-end R&D alone.*"

[20] One may compare the implicit assumption – common to each of the following salient research contribution on endogenous technological change and climate policy analysis -- that innovations resulting from R&D or learning by doing are of the disembodied kind, and therefore their effects are not intermediated by timing and volume of investments in tangible capital formation: Goulder and Schneider (1999), Nordhaus (2002), Bounanno, Carraro and Galeotti (2003), Edenhofer, Carrero and Galeotti (2004), Popp (2004), Lessman, Kemfert et al. (2006), Sue Wing (2006). Rather strikingly, the discussion of these and other items in the literature survey by Gillingham, Newell and Pizer (2008), remains silent on the distinction between embodied and disembodied technological changes, and omits mention of adoption as a determinant of general or energy sector-specific change in productivity or GHG-emissions intensity. While commenting on several aspects of an earlier literature review by Azar and Dowlatabadi (1999), Gillingham et al. (2008) gives no notice of its useful discussion of the evidence documenting the comparatively slow pace of technology diffusion in the energy sector, the responsiveness of private adoption decisions there to changes in performance standards and subsidies, and the role of infrastructure externalities and uncertainty in setting high hurdle rates of return required for lumpy investments.

[21] A recent addition to the theoretical literature by Boucekkine et al. (2012) provides the mathematical foundations for formulating and solving an optimal control resource extraction problem with multiple irreversible ecological regimes.

before that capacity actually can be utilized – a fundamental aspect of the "embodiment" condition – carries the implication that at least two optimal switching moments must be considered, even though only two technologies are involved in the switch.[22] Schumacher (2011) focusses on the timing of a switch towards production based upon a renewable resource that would be induced by the increasing probability of climate disasters under a non-renewable production regime. His analysis, however, posits a production structure in which renewables and non-renewables form a 'complex' of perfectly substitutable inputs that, together with 'generic' capital, produce the economy's aggregate output. Schumacher thus ignores the embodiment of technologies in specific types of capital goods, and suppresses consideration of the need for a sufficient amount of time during which to build up the carbon-free capacity that is required to satisfy future consumption and investment needs.

1.3 Organization of the Paper

The organization of the presentation that follows is straight-forward. Section 2 introduces the basic form and features of the multi-phase optimum control model (in sub-section 2.1) and describes the two ways in which it is extended in this paper. The Basic Model (BAM for short) captures the essential features of the transition problem of an economy that must complete the switch from initial dependence on production facilities that embody a carbon-using technology to producing exclusively with capital that embodies a "carbon-free" technology, i.e., one that enables it to utilize only non-carbon sources of energy. Sub-sections 2.2-2.3 set out the formal structure and the analysis of the optimal dynamics of this transition -- which must be completed before the CO2 emitted in the process has pushed the atmospheric concentration of GHG to the (*TPIRCC*) level that would trigger a catastrophically unstable climate regime. The inter-temporal optimization described in sub-section 2.4 involves solving the "stacked Hamiltonians" (in 2.5) to obtain the durations of the three phases mentioned above (see footnote 22), and the magnitude of the sequenced tangible investments required to build up each of the kinds production facilities, utilize them jointly and eventually shut down un-depreciated fossil-fueled capacity before entering the final phase of sustainable "green" economic growth in a stabilized climate system.

Section 3 sets out the formal structure and analysis of the two models that extend BAM. The introduction of endogenous R&D-driven capital augmenting changes in the carbon-free technology prior to its embodiment and deployment is shown (in 3.1) to result in a modified three-phase model, referred to as "BAM+R&D". A third model, labeled "BAM+R&D+UCL" is obtained (in 3.2) by adding a climate change feedback effect in the form of heightened expected annual rates of damage to the extant capital stock -- driven by the rising atmospheric concentration of CO2, the consequent warming of the earth's surface and increasing moisture in the atmosphere due to the faster evaporation from the oceans' surface, which brings more frequent severe storms, seaboard and riverine flooding and droughts in interior regions. The onset of a higher expected (proportional) rate of "unscheduled capital losses" (UCL) as direct and indirect consequences of damages to reproducible capital, and ecosystem services, is modeled here simply as endogenous jump in the capital stock's rate of technical decay, triggered when cumulative CO2 emissions reach a (joint) weather-systems and ecosystems tipping point. The latter is positioned at an atmospheric concentration level below that of the climate

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[22] In our case too, there must be at least three phases: a pure carbon phase, a mixed carbon and carbon-free phase and a pure carbon-free phase. The simple reason is that capital is a produced means of production, and the very first units of carbon-free capacity must be produced using carbon-based capacity if we start out with a pure carbon phase first.

catastrophe tipping point (*TPIRCC*) which is specified uniformly for BAM and successive extensions of that model.

Section 4 describes the calibration of the three models (in 4.1), and follows that with three sub-sections (4.2 – 4.4) that in turn present the optimal solutions obtained for the transition path of BAM and the two extensions of that model, along with the corresponding results for each version of some parameter sensitivity experiments. The paper concludes in section 5 with a summary of the salient finding and comments on the broader insights offered by this approach to climate policy design, and its limitations that point to priorities for further work.

2   The Basic Model and Its Extended Versions

2.1 Introduction

We use the simplest possible endogenous growth setting in which there are two broad classes of (linear) technologies. One of them is an established technology (called the A-technology) with a relatively high productivity of capital that uses carbon-based energy and that produces CO2 emissions in the process. As stated, these emissions add to the stock of GHG's and so affect the probability of the world getting into a situation of runaway global warming and catastrophe in the end. The alternative technology (further called the B-technology) does not generate CO2 emissions, but has a relatively low capital productivity that needs to be further developed (through R&D) and scaled up (through investment in physical capital) to a level in which it can contribute significantly to the consumption needs of the population at large.

The most elementary setting for growth models that allow formal characterization of the foregoing technological options is provided by models of the 'AK+BK' form, where AK represents the output potential of capital embodying the carbon-based technologies, and BK represents the corresponding capacity of the capital stock embodying technologies permitting reliance on renewable, non-carbon energy sources. In this framework, the variables A and B denote the respective average productivities of the two kinds of capital goods. We proceed by formulating a sequential Hamiltonian system that describes three distinct phases in the transition from carbon-based to carbon-free production. In the first phase, called the business-as-usual phase (BAU for short), only the already existing AK production capacity is active, and in the Basic Model it is assumed that during the BAU phase the technical innovations needed to create production capacity of the BK kind are available but have not been implemented by capital formation of the kind that would be required, because the latter would be less productive than capital that embodies the carbon-using technology. In other words, the known alternative technologies would be relatively costly in terms of instantaneous consumption possibilities foregone, and that disadvantage is not perceived to be offset by being carbon-free. As cumulative CO2 emissions grow with the continuing use of the A technology, the latter becomes ever less (socially) advantageous vis-à-vis the option to build and substitute capital embodying the B technology.

Active utilization of a technology implies two things. First, the basic features of such a technology must be known, while secondly these features are embodied in new capital goods. A technology can be de-activated by not using the capital goods embodying that technology anymore. Because capital goods are technology specific, this implies that a new technology can take over from an old one only by actively investing in the capital goods that implement the new technology and by switching production from the old to the new technology.

In such a setting characterized by linear production technologies and linear cost functions, it can be shown that it is not optimal to invest in both technologies at the same time, if these technologies differ with respect to their net productivity (i.e. net of depreciation). The reason is that a unit of investment for both technologies would represent the same marginal cost in terms of consumption forgone, and so the technology with the highest net marginal product, would generate the highest net marginal welfare gain. Hence, if investment in the A technology is taking place, then investment in the B-technology will be zero and the other way around. We will also allow for the possibility that there is a phase in which no investment in A takes place, even though the existing capital stock is used to produce output while investment in and production using the B-technology is happening at the same time. This second phase will therefore be called the joint production phase (JPR for short). In the final phase, only investment in and production with the B-technology occurs, while the A-technology capital stock has been deactivated at the beginning of that phase. This is the carbon-free phase (CFR for short).

The effects of the production and investment activities during the separate phases distinguished in the model can be summarized as in Figure 1. TBAU, TJPR and TCFR mark the moments in time at which the BAU, JPR and CFR phases begin. In the BAU phase (which starts at t=TU=0), the cumulative emissions (labeled E) are increasing exponentially as the stock of carbon-based capital is growing. In the JPR phase investment in technology A stops, and output using technology A (i.e. Y^A) is at its maximum level but starts to decrease over time, because of technical decay. The stock of technology B capital is built up from scratch starting with the arrival of the JPR phase at t=TJ. Production using technology B (i.e. Y^B) is at full capacity and exponentially increasing during the JPR phase. Cumulative emissions are still increasing, but at a decreasing rate as the stock of carbon-based capital is run down. During the JPR phase, total output is still growing, but at a slower rate than the growth of Y^B. Phase CFR starts when cumulative emissions E hit (at t=TF) the cumulative emissions threshold at $\bar{E}$, just below the climate catastrophe tipping point. During the final phase, only investments capital goods embodying technology B are possible and carbon-based production must stop completely even though there is un-depreciated capacity of the latter type. That shut-down causes a drop in the level of output, which is shown in Figure 1 to jump discontinuously from point I to point II before resuming its growth from the latter level during the CFR phase.
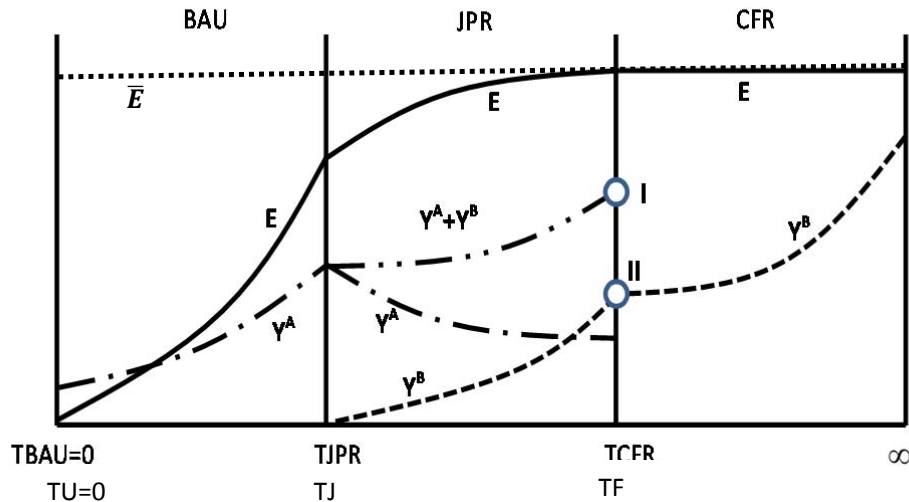

Figure 1. Transition Phases

Clearly, the various phases in the model are seen to be qualitatively different. In the first (BAU) phase, the high productivity carbon-based capital is use to rapidly build more of that production capacity, in order to produce new capital embodying the alternative technology. But, unfortunately, that quickly raises the stock of cumulative CO2 emissions towards its admissible the upper boundary. Before cumulative emissions reach $\bar{E}$, however, phase JPR must have seen carbon-free productive capacity brought to a level that will allow a shut-down of carbon-based production that doesn't force too punishing a drop in consumption. The latter is implied by our assumption that the social planning agent has internalized consumers' relative aversion to negative consumption shocks (i.e., their representative "felicity function" has a positive coefficient of relative risk aversion, i.e., $\theta > 0$), and therefore the optimization of social welfare seeks to smooth changes in the level of consumption over time as much as possible, given the macroeconomic and climate system constraints. From TF onwards the world economy is "green" (having entered the CRF phase), and will have to grow at a relatively slower pace due to the higher unit cost of capital embodying the carbon-free technology. .

In addition to the Basic Model, the follow section formulates two models that extend it in two directions, sequentially incorporating, first, another technology policy option and, next, introducing an adverse feedback effect from the climate system. In the second version of the transition model that will be analyzed, the productivity of capital embodying the carbon-free technology can be raised through (endogenously determined) R&D expenditures before undertaking the capital formation that is necessary to deploy it in production. The third transition model combines the endogenous R&D features of the second version and a constraint that will be imposed on the growth of productive capacity when cumulative CO2 emissions pass a threshold level that causes the global environment to tip into a phase characterized by higher expected rates of weather-related physical damages that result in "unscheduled losses" of services from the global stock of capital. The latter regime-shift is modeled simply as an endogenously timed one-time jump in the total annual rate of scheduled physical depreciation plus "unscheduled capital losses" (UCL). Being anticipated, this impending state change will be reflected in the current shadow price of CO2 emissions, and so feeds back to affect the (optimized_ allocation of tangible and intangible investments prior to the systems' arrival at the high-damages tipping point.

2.2 Phase structure of the Basic Model:

The endogenous growth framework of BAM borrows heavily from the AK-model by Rebelo (1991). However, contrary to the original AK-setting, we distinguish between two types of capital: carbon-based, or black, capital further denoted by K_A and carbon-free, or green, capital further denoted by K_B. The capital stocks in this model are subjected to exponential decay at rates δ_x, for $x \in \{A, B\}$. Because of the linearity of the production functions in an AK-setting, and since one unit of capital takes one unit of consumption foregone for the two technologies distinguished, it follows that there will always by investment in just one type of capital at the time. Hence, gross investment in a particular technology is either equal to zero, or it is equal to total savings. Welfare in this setup comes from consumption only, and we use the CIES inter-temporal welfare function to describe the total flow of welfare over time. The activities during the different phases are summarized in Table 1.

Activities	BAU Phase	JPR Phase	CFR Phase
Investment	$I_A > 0$	$I_B > 0$	$I_B > 0$
Production	$Y_A = A.K_A$	$Y_A = A.K_A$ $Y_B = B.K_B$	$Y_A = 0$ $Y_B = B.K_B$
Capital Accumulation	$\dot{K}_A = Y_A - \delta_A.K_A - C$	$\dot{K}_A = -\delta_A.K_A$ $\dot{K}_B = Y_A + Y_B - \delta_B.K_B - C$	$\dot{K}_B = Y_B - \delta_B.K_B - C$
CO2 Emissions	$\dot{E} = \varepsilon_A.K_A$	$\dot{E} = \varepsilon_A.K_A$	$\dot{E} = 0$

Table 1. BAM activities

In this table, I_x refers to the amount of investment in capital of type x, where $x \in \{A, B\}$ refers to 'Carbon-based' capital and carbon-free capital, respectively. Similarly, Y_x refers to the flow of output using K_x, where K_x is the stock of capital of type x. We also assume that Y_x is proportional to K_x with a constant productivity of capital as a factor of proportion. Typically, we use A and B to denote the capital productivities of technologies A and B, implying that $Y_A=A.K_A$, etcetera. Finally, the instantaneous flow of CO2 emissions is proportional to the capital stock in use with a constant factor of proportion ε_x for $x \in \{A, B\}$. Obviously, $\varepsilon_B = 0$. Note that when production on some type of carbon-based capital ceases, this very fact initiates another phase, since this introduces a difference between the composition of activities between the JPR and CFR phases. So $Y_A=0$ implicitly defines the arrival time of the CFR phase, and the scrapping of carbon-based capital at t=TF.

2.3 BAM: The inter-temporal optimization setting

It should be noted that the fact that the final phase of BAM is a pure AK-setting, allows us to obtain the optimum consumption path for the CFR phase directly, given an initial value for K_B^{TF}.[23] The welfare generated during the CFR phase depends therefore on the terminal value of K_B at the end of the JPR phase. It follows that the distribution of a state variable over the entire path is optimal when the marginal costs of having to deliver an extra unit of the state variable in its role as a terminal value at some time t* is exactly matched by the marginal benefits that this extra unit of the state variable generates as the initial value for the optimum continuation from t*. Since these marginal benefits and marginal costs are captured by the co-state variables (see Leonard and Van Long (1992: Ch. 4), the latter need to be continuous along an optimum path: states and co-states don't jump. [24]

An optimum path can be thought of as a combination of an optimum first step and an optimum continuation (as in dynamic programming problems), which allows us to interpret our multiphase transition model as a finite horizon optimum control problem with a free endpoint and a scrap value function, as described in Leonard and Van Long (1992: Ch. 7), hereafter L&VL: ch.7). This is the situation that is of direct relevance in our case, since we do not know on beforehand when the next phase will start. However, on an optimum path, postponing or extending a particular phase by an infinitesimal amount of time shouldn't change the valuation of the entire path. L&VL show that the derivative of the value function (in our case the present value of total welfare) with respect to the terminal date (of a phase) matches the value of the Hamiltonian at that date. This makes sense, as the Hamiltonian at some moment in time measures the contribution to the value function of the optimal use of all resources available at

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[23] See, Barro and Sala-i-Martin (2004), chapter 4 in particular.

[24] In the case of pure state-constraints, however, co-states can jump (cf. Leonard and Van Long (1992: Ch. 10).

that moment in time. But in a sequence of phases, lengthening the one phase by a unit of time implies shortening the next phase by the same unit of time. So we should keep on postponing the arrival of the next phase as long as the Hamiltonian of the earlier phase exceeds that of the later phase. The optimum switching moment between any two phases is therefore implicitly defined by the requirement that the Hamiltonians of two adjoining phases must be the same when evaluated at the moment of the phase-change. Again, this makes perfect economic sense, as the value of the Hamiltonian at the end of the current phase can be seen as the benefits of expanding the current phase by a marginal unit of time, while the Hamiltonian at the beginning of the next phase can be seen as the opportunity cost of that expansion. In practice, the equality of the Hamiltonians evaluated under the conditions relevant in either of the phases just before and just after a phase change, will result in a condition that needs to be met by a set of states and co-states evaluated at the moment of the phase-change.

The differences in the nature of the activities at the moment of a phase change can be used to implicitly describe the conditions that should be met at the moment of a phase change. For example, at t=TF it must be the case that carbon-based capital is deactivated. For t>=TF it must therefore be the case that the shadow price of carbon-based capital is zero, since the cumulative emission threshold has been reached and carbon-based capital can therefore not be used anymore and has become worthless from then on.

2.4 BAM: Formal description of the optimal 3-phase transition path

The overall welfare function consists of a summation of integral welfare derived from the flow of consumption during the three phases distinguished in BAM:

$$W_0 = \int_0^{TJ} \frac{e^{-\rho \cdot t} \cdot (C_t)^{1-\theta}}{(1-\theta)} dt + \int_{TJ}^{TF} \frac{e^{-\rho \cdot t} \cdot (C_t)^{1-\theta}}{(1-\theta)} dt + \int_{TF}^{\infty} \frac{e^{-\rho \cdot t} \cdot (C_t)^{1-\theta}}{(1-\theta)} dt \ . \tag{1}$$

In equation (1) W_0 measures the present value of total welfare at time t=0, at which, by assumption, phase U begins. In equation (1), ρ is the rate of discount, while $1/\theta$ is the (constant) intertemporal elasticity of substitution. C_t is consumption at time t, while TJ and TF are the moments in time at which the JPR and CFR phases begin. Given the exposition on inter-temporal optimization above, the time paths that would maximize (1) can be obtained by solving the time paths for the Hamiltonian problems that can be defined for the individual phases, while linking those time paths together by means of the requirements of optimum phase lengths (implying the equality of the Hamiltonians for phases U and J at t=TJ and for the JPR and CFR phases at t=TF). Effectively this comes down to maximizing (1) w.r.t. the flows of consumption during each phase and w.r.t. the phase-lengths themselves, constrained by the technologies that are relevant in each phase, by the stocks inherited from previous phases, and by the thresholds that are relevant during the various phases. We will now solve the Hamiltonian problems for each individual phase.

The BAU phase

Using the superscripts U (for BAU), J (for JPR) and F (for CFR) to denote the phase to which a particular variable pertains, while dropping the time subscript for ease of notation, the present value Hamiltonian H^U is given by:

$$H^U = \frac{e^{-\rho \cdot t} \cdot (C^U)^{1-\theta}}{(1-\theta)} + \lambda_{K_A}^U \cdot \left((A - \delta_A) \cdot K_A^U - C^U \right) + \lambda_E^U \cdot \varepsilon_A \cdot K_A^U \ . \tag{2}$$

In equation (2), C is the only control variable, while K_A and E are the state variables and $\lambda_{K_A}^U$ and λ_E^U are the corresponding co-states. As first order conditions we have:

$$\frac{\partial H^U}{\partial C^U} = e^{-\rho \cdot t} \cdot (C^U)^{-\theta} - \lambda_{K_A}^U = 0 \Rightarrow C^U = \left\{e^{\rho \cdot t} \cdot \lambda_{K_A}^U\right\}^{-1/\theta} \tag{3}$$

$$\frac{\partial H^U}{\partial K_A^U} = \lambda_{K_A}^U \cdot (A - \delta_A) + \lambda_E^U \cdot \varepsilon_A = -\dot{\lambda}_{K_A}^U \tag{4}$$

$$\frac{\partial H^U}{\partial E^U} = 0 = -\dot{\lambda}_E^U \tag{5}$$

$$\frac{\partial H^U}{\partial \lambda_{K_A}^U} = \dot{K}_A^U = \left((A - \delta_A) \cdot K_A^U - \left\{e^{\rho \cdot t} \cdot \lambda_{K_A}^U\right\}^{-1/\theta}\right) \tag{6}$$

$$\frac{\partial H^U}{\partial \lambda_E^U} = \dot{E}^U = \varepsilon_A \cdot K_A^U \tag{7}$$

Equation (6) is obtained by means of substitution of equation (3) into the macro-economic budget constraint which states that output is used for consumption and (gross) investment purposes. Equations (4)-(7) constitute a simultaneous system of differential equations that can be solved forward in time, given a set of initial values for the various state and co-state variables. This will give rise to terminal values of those same state and co-state variables at the terminal date of phase U, i.e. at t= TJ, the value of which is unknown so far.

The JPR phase
The JPR phase differs from the BAU phase since investment in the A technology has stopped and that in the B-technology begins. However, the carbon-based capital stock K_A is still used for production purposes. The present value Hamiltonian for the JPR phase, i.e. H^J, is now given by:

$$H^J = \frac{e^{-\rho \cdot t} \cdot (C^J)^{1-\theta}}{(1-\theta)} + \lambda_{K_A}^J \cdot \left(-\delta_A \cdot K_A^J\right) + \lambda_{K_B}^J \cdot \left((B - \delta_B) \cdot K_B^J + A \cdot K_A^J - C^J\right) + \lambda_E^J \cdot \varepsilon_A \cdot K_A^J \quad . \tag{8}$$

As in the BAU phase, we have just one control, i.e. C^J, but three states K_A, K_B and E and corresponding co-states $\lambda_{K_A}^J$, $\lambda_{K_B}^J$ and λ_E^J. As first order conditions we have:

$$\frac{\partial H^J}{\partial C^J} = e^{-\rho \cdot t} \cdot (C^J)^{-\theta} - \lambda_{K_B}^J = 0 \Rightarrow C^J = \left\{e^{\rho \cdot t} \cdot \lambda_{K_B}^J\right\}^{-1/\theta} \tag{9}$$

$$\frac{\partial H^J}{\partial K_A^J} = \lambda_{K_A}^J \cdot -\delta_A + \lambda_{K_B}^J \cdot A + \lambda_E^J \cdot \varepsilon_A = -\dot{\lambda}_{K_A}^J \tag{10}$$

$$\frac{\partial H^J}{\partial K_B^J} = \lambda_{K_B}^J \cdot (B - \delta_B) = -\dot{\lambda}_{K_B}^J \tag{11}$$

$$\frac{\partial H^J}{\partial E^J} = 0 = -\dot{\lambda}_E^J \tag{12}$$

$$\frac{\partial H^J}{\partial \lambda_{K_A}^J} = \dot{K}_A^J = -\delta_A \cdot K_A^J \tag{13}$$

$$\frac{\partial H^J}{\partial \lambda_{K_B}^J} = \dot{K}_B^J = (B - \delta_B) \cdot K_B^J + A \cdot K_A^J - \left\{e^{\rho \cdot t} \cdot \lambda_{K_B}^J\right\}^{-1/\theta} \tag{14}$$

$$\frac{\partial H^J}{\partial \lambda_E^J} = \dot{E}^J = \varepsilon_A \cdot K_A^J \tag{15}$$

Equation (14) is again obtained by means of substitution of optimum consumption levels (as given by equation (9)) into the macro-economic budget constraint. As before, equations

(10)-(15) constitute a simultaneous system of differential equations that can be solved forward in time, given initial values for the various state and co-state variables. Note that the initial values in the JPR phase for those state and co-state variables that both systems have in common, are the same as the terminal values for those variables at the end of the BAU phase because of the continuity of state- and co-state variables along an optimum path. For a given value of TF, this system of differential equations allows the forward calculation of terminal values for the states and co-states in the JPR phase at time t=TF, which will then function as the initial values for the states and co-states during the carbon-free phase.

The CFR phase

Phase F differs from phase J in that the carbon-based capital stock is discarded, and consequently the flow of CO2 emissions drops to zero. From t=TF production is totally green. The present value Hamiltonian for phase F, i.e. H^F, is now given by:

$$H^F = \frac{e^{-\rho \cdot t} \cdot (C^F)^{1-\theta}}{(1-\theta)} + \lambda_{K_B}^F \cdot \left((B - \delta_B) \cdot K_B^F - C^F\right) + \lambda_E^F \cdot 0 \tag{16}$$

As in the BAU and JPR phase, we have just one control, i.e. C^F, but two states K_B and E and the corresponding co-state variables $\lambda_{K_B}^F$ and λ_E^F. As first order conditions we have:

$$\frac{\partial H^F}{\partial C^F} = e^{-\rho \cdot t} \cdot (C^F)^{-\theta} - \lambda_{K_B}^F = 0 \Rightarrow C^F = \left\{e^{\rho \cdot t} \cdot \lambda_{K_B}^F\right\}^{-1/\theta} \tag{17}$$

$$\frac{\partial H^F}{\partial K_B^F} = \lambda_{K_B}^F \cdot (B - \delta_B) = -\dot{\lambda}_{K_B}^F \tag{18}$$

$$\frac{\partial H^F}{\partial E^F} = 0 = -\dot{\lambda}_E^J \tag{19}$$

$$\frac{\partial H^F}{\partial \lambda_{K_B}^F} = \dot{K}_B^F = (B - \delta_B) \cdot K_B^F - \left\{e^{\rho \cdot t} \cdot \lambda_{K_B}^F\right\}^{-1/\theta} \tag{20}$$

$$\frac{\partial H^F}{\partial \lambda_E^F} = \dot{E}^F = 0 \tag{21}$$

As before, equation (20) is obtained by substituting equation (17) into the macro-economic budget constraint. Again, equations (18)-(21) constitute a simultaneous system of differential equations that can be solved forward in time, given initial values for the various state and co-state variables inherited from phase J. However, in this case the terminal values for the states and co-states are implicitly described by the standard transversality conditions in an AK setting that require the present value of the carbon-free capital stock to approach zero at the terminal date, i.e. in this case at time infinity. For cumulative emissions, the terminal value had already been reached at t=TJ, when cumulative emissions hit the threshold.

Transversality conditions

For the CFR phase the standard transversality condition (further called TVC for short) applies regarding the value of carbon-free capital at time infinity:

$$\lim_{t \to \infty} \lambda_{K_B,t}^F \cdot K_{B,t}^F = 0 \tag{22}$$

where we have now added time-subscripts. Note that (18) can be integrated directly to obtain the time path for $\lambda_{K_B,t}^F$ which can then be substituted into (20) to obtain the time path for $K_{B,t}^F$. This yields:

$$\lambda_{K_B,t}^F = \lambda_{K_B,TF}^F \cdot e^{-(B-\delta_B)\cdot(t-TF)} \tag{23}$$

$$K_{B,t}^F = \frac{e^{-\frac{t\rho-(t-TF)(B-\delta B)}{\theta}}\theta\cdot\left(\lambda_{K_B,TF}^F\right)^{-1/\theta}}{\rho+(\theta-1)(B-\delta B)} + e^{(t-TF)(B-\delta B)}(K_{B,TF}^F - \frac{e^{-\frac{TF\rho}{\theta}}\theta\cdot\left(\lambda_{K_B,TF}^F\right)^{-1/\theta}}{\rho+(\theta-1)(B-\delta B)}) \tag{24}$$

Equations (23) and (24) can be substituted into TVC (22), and we find that in order for TVC (22) to hold the following conditions must be satisfied:

$$\frac{\rho+(\theta-1)(B-\delta_B)}{\theta} > 0 \tag{25}$$

$$K_{B,TF}^F = \frac{e^{-\frac{TF\rho}{\theta}}\theta\cdot\left(\lambda_{K_B,TF}^F\right)^{-1/\theta}}{\rho+(\theta-1)(B-\delta B)} \qquad . \tag{26}$$

When substituting (26) into (24), we find that:

$$K_{B,t}^F = K_{B,TF}^F \cdot e^{\frac{(B-\delta B-\rho)(t-TF)}{\theta}} \qquad . \tag{27}$$

It follows from (27) that if the structural parameters are such that (25) is met and if we pick consumption at time T_F (hence $\lambda_{K_B,TF}^F$ (see equation (17)) such that (26) is met, then the carbon-free capital stock will grow at the steady state growth rate $(B-\delta B-\rho)/\theta$ from time t=TF.

Apart from the TVC above, we require that:

$$\lambda_{K_A,TF}^J = 0 \qquad , \tag{28}$$

which sets the shadow price of carbon-based capital to zero at the end of the JPR phase (i.e., at the moment it is discarded), because an extra unit of capital added then would not produce anything and, being useless, would be worth nothing.

Lastly, there are two TVCs that pertain to the optimum length of the BAU and the JPR phase, and that require the equality of the Hamiltonians of the various phases at different points in time. For the optimum length of the BAU phase (given by the value of TJ, since TU=0 by assumption), we must have $H_{TJ}^U = H_{TJ}^J$ [25], whereas the optimum start date for the F phase is determined by the requirement that $H_{TF}^J = H_{TF}^F$. Using the definitions of the Hamiltonians in (2), (8) and (16), as well as imposing the F.O.C. regarding consumption in (3), (9) and (17) together with the continuity constraints on states and co-states that feature in both adjoining phases, we obtain implicit descriptions of the arrival dates of the JPR and CFR phases. These are given by:

$$\lambda_{K_A,TJ}^J = \lambda_{K_B,TJ}^J \tag{29}$$

$$A \cdot \lambda_{K_B,TF}^J = -\varepsilon_A.\lambda_{E,TF}^J \qquad . \tag{30}$$

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[25] H_t^P is short for the value of the Hamiltonian pertaining to phase P at time t. In the equality $H_t^P = H_t^{P+1}$ it follows that the time-coordinate of H_t^P cannot be exactly the same as that of H_t^{P+1} since P+1 is the phase coming directly *after* phase P. Consequently, $H_t^P = H_t^{P+1}$ must be read as $\lim_{\varepsilon\downarrow 0} H_{t-\varepsilon}^P = H_t^{P+1}$, but for simplicity we stick to our original notation.

Equation (29) states that investment in the carbon-based technology should stop at the moment that the shadow price of the A technology is equal to (and thereafter drops below) the shadow price of the B technology. Since the marginal cost of obtaining a unit of capital is the same in both cases (i.e. one unit of consumption foregone), equation (29) is consistent with the maximization of the (present value) welfare surplus associated with investment. Equation (30) states that production using carbon-based capital should stop the moment that the benefits from continuing to use a unit of capital (the LHS of equation (30), as one unit of capital produces A units of output, and each unit of output is worth $\lambda^J_{K_B,TF}$ in present value welfare terms at t=TF) matches the cost of doing that (as given by the RHS of equation (30), since one unit of capital produces ε_A units of CO2 emissions, each at a cost of $-\lambda^J_{E,TF}$ [26]). Equation (30) is therefore consistent with the zero quasi-rent condition as an implicit description of the optimum moment to scrap existing capacity known for a long time from the clay-clay vintage literature. [27]

2.5 BAM: Sequential numerical solution of the differential equation systems

The three systems of differential equations, further called SU, SJ and SF, can in principle be solved, as the initial and terminal values we have available for the state variables, and the TVCs that provide either some fixed points for the time paths of the co-states (cf. equation (28)), or link the co-states to a state-variable which time path has been fixed through a given initial value (cf. equation (26)), or that links different co-states at some point in time (cf. equations (29) and (30)) provide exactly enough information to obtain a fixed point for all time paths concerned. To see this, it should be noted that we need to obtain fixed points for the time paths for three different state variables (KA,KB and E), and for their corresponding co-state variables, as well as the optimum values of the phase lengths of phases U and J. Hence, for BAM we need 8 pieces of information. We have initial values available for $K^U_{A,0} = \bar{K}^U_{A,0}$, $K^J_{B,TJ} = 0$, and $E^U_0 = \bar{E}^U_0$. In addition, we have a terminal value for cumulative emissions:$E^F_{TF} = \bar{E}^F_{TF}$. The transversality conditions given by equations (26), (28), (29) and (30) then provide the remaining information that is needed.

A numerical solution of SU can easily be obtained, conditional on some a priori values of $\lambda^U_{K_A,0}$, λ^U_E and TJ, given the initial values of $K^U_{A,0}$ and E^U_0. The solution of SU then provides terminal values for $\lambda^U_{K_A,TJ}$ and $\lambda^U_{E,TJ}$ that, on account of the continuity of states and co-states give rise to initial values for SJ, since we must have that $\lambda^U_{K_A,TJ} = \lambda^J_{K_A,TJ}$, $\lambda^U_{E,TJ} = \lambda^J_{E,TJ}$, $K^J_{A,TJ} = K^U_{A,TJ}$, $K^J_{B,TJ} = 0, E^J_{TJ} = E^U_{TJ}$. We only need an initial value for $\lambda^J_{K_B,TJ}$ as well as an a priori value for TF to be able to calculate SJ forward in time. That initial value is provided by the TVCs listed in equation (29). Given the a priori values for $\lambda^U_{K_A,0}$, λ^U_E , TJ and finally TF, we are able to calculate the terminal values for $\lambda^J_{K_B,TF}$, $\lambda^J_{E,TF}$, $K^J_{B,TF}$ and E^J_{TF} , which, again using the requirement of the continuity of states and co-states, imply that $K^F_{B,TF} = K^J_{B,TF}$, $E^F_{TF} = E^J_{TF}$, $\lambda^F_{K_B,TF} = \lambda^J_{K_B,TF}$ and $\lambda^F_{E,TF} = \lambda^J_{E,TF}$. The time paths thus obtained for all states and co-states, in combination with our guesses of the various phase lengths, can now be used to evaluate the differences between the

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[26] Note that the shadow price of emissions itself is negative, since an additional unit of cumulative emissions would reduce potential welfare rather than increasing it.

[27] See, e.g., Johansen (1959) and Solow et al. (1966). More recently, Boucekkine (2011) also used this condition, which states that a vintage once installed should be discarded as soon as it's quasi-rents become negative, a rule that holds in this case since the quasi-rents consist of the social welfare value of discounted output less total variable CO2 emissions costs.

RHS and the LHS of the terminal constraint $E_{TF}^{TF} = \bar{E}_{TF}^{TF}$ and those of the TVCs not used so far, i.e. of equations (26), (28) and (30). Obviously, if these constraints were satisfied by the values chosen *a priori* for $\lambda_{K_{A,0}}^{U}$, λ_E^U, TJ, and TF, we would have found the solutions for all the welfare-maximizing time paths. Being the result of a guess, these differences generally will not equal zero at the outset. But, in that case, a search algorithm such as the steepest descent method that we thus far have been using with success – should readily find the set of initial values that will simultaneously satisfy all of the TVCs.[28] It should be noted that a similar approach can be followed for versions of the model with more than 3 phases, as we will describe further below.

3. Extensions of the BAM Framework

This section describes the modifications of the basic 3-phase model that extend the structure of BAM in two ways. The first introduces the option of investment in research and development directed to lowering the unit capital costs of production facilities that embody technologies that do not use fossil fuel as an energy source, and the second takes account of an adverse feed-back effect on the global economy resulting from the warming that accompanies the transition.

3.1 BAM + R&D: The modified model's structure

To introduce R&D investment-driven endogenous technical changes affecting the economic performance of the carbon-free technology, we posit that these will be confined to raising the average (and marginal) productivity of the capital goods that embodies the new technology. The latter lowers the real unit cost of carbon-free production capacity by irreversibly raising *B* above its initial value in the opening (BAU) phase, and doing so prior to the start of the capital formation that is necessary to implement this form of technical change and begin the transition away from an economy reliant upon burning fossil fuels.

Since we are building upon the BAM framework that uses an AK growth model setting, the familiar modeling of endogenous technical change deriving from R&D activities is not readily applicable. The classic exemplars of the later in the literature – found in Lucas (1988), or Romer (1990), or (more implicitly) in Aghion and Howitt (1992) -- specify that the dependence of the proportional rate of change in total input productivity upon the flow of R&D expenditures is described by a simple functional relationship of the following form:

$$\dot{B} = \varsigma \cdot R \cdot B \tag{31}$$

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[28] In fact, given the simplicity of our system, it is possible to obtain analytical solutions for all time paths, even though these are in part quite intricate non-linear expressions involving hyper geometric functions. These integral equations link the various initial and terminal conditions as well as the values of TJ and TF together through a simultaneous system of non-linear equations. Using that system, we were able to find the fixed points for all time paths by numerically solving the non-linear system for the fixed points. The time paths for each of the state and co-state variables could then be obtained by substituting the numerical values thus found for the fixed points into the analytical solutions of all time paths involving these fixed points. This procedure proved to work, but is rather tedious and time consuming and it's feasibility depended crucially on the simplicity of the model. Minor deviations from the AK-set-up proved to make using the analytical method infeasible. This provided a strong incentive to use the sequential numerical solution method outlined in the main text. Both methods do generate the same results, as they should, but the sequential numerical method is much more efficient in terms of both obtaining the system of differential equations to be solved, and finding the solution. Nonetheless, more work is needed to improve upon the rather crude steepest descent search routine that we are employing at the moment. The latter converges rather slowly, if at all, for the more intricate versions of the model extensions we have built up to date and which are not reported here. Nonetheless, the numerical model does allow us to expand the model in ways that would make it impossible to solve using the analytical solution approach.

where R represents R&D resources measured here as consumption foregone. But, with the linear, single-factor specification of the AK model's production function, the average (and marginal) productivity of capital remains constant as the economy accumulates capital. No contributions from increases in other factors of production (say, the state of technical knowledge, or human capital) are needed to prevent the capital stock's growth from driving its marginal productivity downwards. Were a specification such as (31) to be introduced in a simple AK-setting (disregarding the embodiment requirement for the sake of simplicity in making this point clear) the resulting growth path of the system would become explosive, rather than one that could attain a steady-state equilibrium rate.[29] Hence, we have opted for a specification of the R&D process that allows for a decreasing marginal product of R&D and a growth rate of B that asymptotically approaches zero for any constant positive allocation of resource to the R&D activity. This assures that the real marginal rate of return from maintaining investment in R&D will diminish over time. The simple form for this function therefore is:

$$\dot{B} = \varsigma \cdot R^{\beta} \cdot (\overline{B} - B) \tag{32}$$

where $\overline{B}$ is the asymptotic value for capital productivity and where $\varsigma > 0$ and $0 < \beta < 1$ are constants.[30] Starting out with a basic technical design that permits productivity to be at B_0, $0 < B_0 < \overline{B}$, the marginal product of R&D investment will be falling as R increases and B rises towards its upper bound.

Several features of this specification argue for its' use in the present applications context. Firstly, it has the advantage of introducing decreasing returns to R&D in a setting that, unlike the conventional endogenous growth models, excludes the possibility of a specific technology being rendered infinitely productive (and so resulting in infinitely rapid growth) merely by the application of more and more massive R&D expenditures at any particular moment in time.[31] Allowing decreasing marginal returns in R&D recognizes that at a given stage in the advance of knowledge the state of fundamental scientific understanding of the physical processes involved may still be inadequate to permit the effective application of more and more resources to the solution of a particular practical problem -- such as the further improvement of the productivity of a particular class of technology-embodying capital facilities.[32] That more

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[29] This follows from the fact that with a constant level of R in equation (31) the rising level of capital's marginal (and average) productivity would increase the attractiveness of engaging in R&D while providing the extra investment resources for that purpose by accelerating output growth.

[30] To keep keep the matter simple for computational purposes in this investigation, we have used R, the current flow of resources as the control variable in the R&D process. But it will be conceptuallymore satisfactory, and should not be computationally problematic in future work with this model to replace R in equation (31) by a latent state variable representing the depreciated stock of knowledge that at time t is germane to this directed R&D activity. Following conventional practice in the empirical literature on productivity effects of R&D, the curent state of knowledge useful for modifying the state variable B, can be indexed by the current value of the integral of depreciated flows of R, added to an initial measure of the knowledge stock available at t= 0 , the commencement date for the activity. This leaves R as the control variable, but wil have the effect of smoothing changes in the growth rate of B.

[31] The equilibrium (steady-state) growth rate in a standard AK-model rises linearly with the productivity of capital (cf. Barro and Sala-i-Martin, 2004), and therefore with the flow rate of R&D expenditures. The existence of technology-specific intrinsic productivity bounds that are set in the limit (by the physical properties of the materials, chemical and electrical processes entailed in production) makes this so implausible that even its assertion as a metaphor is of dubious usefulness . By contrast, a further, technical advantage of the specification given by equation (32) is that the rate of improvement in the productivity of capital is jointly concave in B and R, assuring fulfillment of a necessary condition for the welfare maximization problem to have a solution.

[32] Conceptually, this formulation of the effects of investment in R&D activities may be thought to reflect a Platonic world in which a finite number of solution possibilities for technical transformations are present from the start of

restrictive view of the transformative power of investment in R&D is appropriate also because the concern in this context is not with the undirected global expansion of the technological opportunity set typically envisaged in theoretical growth models. Rather, the aim of the "directed R&D" in the present model is to enhance the economic properties of particular kinds of process inventions, with new product inventions only insofar as alterations in product characteristics are consequential for raising the efficiency of capital inputs into carbon-free production processes.[33]

Introducing (32) into the Basic Model requires a number of modifications: the first of these extends the Hamiltonian by adding the value of increases in B due to R&D. Next, the macro-economic budget constraint that describes the accumulation of capital must be adjusted to reflect the fact that a unit of resource inputs in R&D, R, requires allocation of a unit of aggregate output (i.e., foregoing its use in consumption or tangible investment). Instead of just the one control variable C, we now have an additional control variable R, and an additional state variable, i.e. B (the carbon-free productivity parameter of BAM). B has become a state variable since it can change as long as R&D is taking place, but must remain constant after R&D has ceased. Leonard and Van Long (1992: Ch.7) describe how the latter situation can be handled. Their approach is to regard the Hamiltonian as a function of B, i.e. *H(B)*, while substituting $\dot{B} = 0$ as the dynamic constraint on B during the phases when investments in R&D are not made (and *B* therefore will not be changing). In all phases we have that the equation of motion for the co-state variable associated with B is given by $\dot{\lambda}_B = -\partial H / \partial B$. The reason for explicitly specifying the dynamic constraints on the co-state for B during the phases where B itself is not changing (because R&D has halted) is that the terminal value of B (when the R&D process stops) will influence the generation of welfare in the following phases. Thus, through the continuity of co-states and states, information about the future welfare effects of having a high value of B in the JPR and CFR phases, can influence allocation decisions during the phase with positive R&D that would directly involve the trade-off between R&D and other uses of output, like consumption and tangible capital investment.

Note that it can be shown that once an initial value for B_0 is available, then it pays not to wait to improve the productivity of capital embodying carbon-free technology by engaging in R&D (see Appendix A). Hence, without loss of generality, we may assume that the basic idea underlying the carbon-free technology is available from t=TU=0. In that case, R&D should start at t=TU, and it should stop at t=TJ, that being the moment at which investment in the formation of carbon-free production capacity commences. Consequently, the BAM+R&D model has the same number of phases as BAM. For the BAU phase, assuming that R&D is done from the very beginning, the three following equations have to be added[34]: $\dot{\lambda}_B = -\partial H / \partial B$,

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time, but these as a rule will not reveal themselves spontaneously. They can be uncovered, however, and formulated for practical application through costly research and development procedures based upon the existing state of fundamental scientific knowledge, rather than being created *de novo* and without limit by the expenditure of resources in the performance of R&D activities

[33] Following this interpretation, adding endogenous technological change to the Basic Model allows us to characterize the optimal path of *global R&D that is directed to increasing the productivity of green capital*. Correspondingly, the impact of R&D investment on economic welfare is modeled as being felt indirectly, rather than directly in the form of pure product quality enhancements. In other words, the welfare gains come through reduction of the sacrifice of consumption utility required in the transition, and for the subsequent sustained growth of (per capita) consumption under stabilized climatic conditions.

[34] Note that for ease of notation we have dropped the time subscript and the phase superscript except where the presence of the time subscript is needed.

$\dot{B} = \varsigma \cdot R^{\beta} \cdot (\bar{B} - B)$ and $\partial H / \partial R = 0$, and the capital accumulation constraint takes the correspondingly modified form: $\dot{K}^{A} = (A - \delta^{A}).K^{A} - C - R$.

Furthermore, the revised Hamiltonian for the BAU phase is now given by: $H = e^{-\rho \cdot t} \cdot C^{1-\theta} /(1 - \theta) + \lambda_{A} \cdot \{(A - \delta_{A}) \cdot K_{A} - C - R\} + \lambda_{B} \cdot \dot{B}$.

Since there is now an additional state variable, there is also an additional TVC. As usual, we require that at infinity the present (utility) value of B should approach zero, i.e. $\lim_{t \to \infty} \lambda_{B,t} \cdot B_{t} = 0$. However, $B_{t} = B_{TJ} \quad \forall t \geq TJ$, since R&D has ceased at t=TJ which has turned B into a constant for $t \to \infty$. Consequently, the additional TVC is reduced to the requirement that $\lim_{t \to \infty} \lambda_{B,t} = 0$. Using (25), it can be shown that the latter TVC implies:

$$\lambda_{B,TF}^{F} = K_{B,TF}^{F} \cdot \lambda_{K_{B,TF}}^{F} \cdot \frac{\theta}{(B_{TF}^{F} - \delta_{B}) \cdot (\theta - 1) + \rho} \ . \tag{33}$$

It is hard to give an interpretation for equation (33), and *a fortiori*, for the revised version of the transversality condition that now determines the optimum length of the BAU phase. That TVC differs from the corresponding condition in BAM, since the R&D process is active during the BAU phase and inactive during subsequent phases. Consequently, the Hamiltonians of the BAU and JPR phase evaluated at t=TJ, will now involve terms coming from the R&D function as well as the corresponding co-state evaluated at t=TJ, resulting in a complicated expression linking the various states and co-states together at t=TJ. Because it cannot readily be interpreted, the expression is not given here. Note that (33) in combination with the initial value for B, i.e. B_0, provide enough information to solve this revised system of differential equations.

3.2 Allowing for warming-driven "Unscheduled Capital Losses": BAM+ R&D+UCL

The final version concerns a combination of BAM and R&D that includes the introduction of a "tipping point" beyond which further warming, driven by the rising concentration of CO_2 in the atmosphere, will bring a higher expected annual rate of "unscheduled losses" of productive capacity due to the direct and indirect effects of damages to the carbon-based capital stock -- as a result of more frequent extreme-weather events, and more extensive coastal and riverine flooding, and more prolonged drought in interior regions.[35]

For present purposes we have modeled this in the following (simplest) way, supposing that there is a threshold that when crossed will trigger the onset of a "high damage" regime, and defining that "extreme-weather tipping point" to be a critical level of cumulative CO_2 emissions that is reachable within the BAU phase. A step-function rise in the proportional annual rate of physical losses of capital services, in effect a jump in the decay parameter from the normal physical rate of depreciation, takes place when that tipping point is reached, thereby splitting the BAU phase in to an initial "low damage" sub-phase and the subsequent sub-phase of "high

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[35] The restriction of expected annual "unscheduled losses" of productive capacity to the carbon-using capital stock, implicitly assumes, firstly, that the inherited energy, transport and communications infrastructures, and areas of industrial concentration in the BAU phase reflected locational choices made in an era before the greater vulnerability of those regions to extreme-weather due to global warming was foreseen. Secondly, a form of adaptation is implied in the (optimistic) assumption that the subsequently formed carbon-free capital stock has been "defensively" designed and/or located in this respect –some portion of the directed R&D activities, and the actual deployment cost, reflecting the incremental costs of thereby protecting its expected marginal social rate of return from being reduced by high physical damanges and temporary "outages". See also Brock et al.(2012) for recent modeling of regional climatic changes due the geographic variations in warming: these should be considered in modeling damage levels and defensive adaptation needs.

damages". All the phases following the latter part of the BAU phase are also characterized by continuation of the "high damage" regime.[36] Once again, this is a situation that is described in Leonard and Van Long (1992: Ch.10), where the regime shift from low damages to high damages initiated by a state variable hitting a particular threshold, implies a jump in the corresponding co-state variable.

Since the rate at which CO2 emissions are accumulated depends on production decisions, the moment in time at which the damage threshold will be hit, depends on these very production decisions too. Hence, the arrival time of the high weather related damage sub phases is subject to choice and therefore to optimal decision-making.[37] Consequently, we can optimally choose the moment at which the high damage sub-phase will arrive. We can implement this again by requiring that the Hamiltonians evaluated at the moment of arrival of the high damage sub phase will be the same immediately before and after its arrival. This implies that this model will have four different phases instead of three. It follows that we need to determine an extra phase length in addition to the size of the jump in the co-state for cumulative emissions at the time of arrival of the first high damage sub-phase. In addition to the given initial and terminal values of the BAM+R&D model as well as the corresponding transversality conditions, we have an additional terminal value in the form of the location of the damage threshold itself, in combination with the requirement of the equality of the Hamiltonians at both sides of the damage phase change. It turns out that the transversality condition for the arrival of the high damage sub phase during the BAU phase is given by:[38]

$$\lambda_{KA,TUH}^{U} \cdot (\delta_A^H - \delta_A^L) = \epsilon_{\mathrm{A}} \cdot \left(\lambda_{E,TUH}^{U,H} - \lambda_{E,TUH}^{U,L} \right) \quad . \tag{34}$$

Equation (34) should hold exactly at the arrival time of the high damage sub-phase, i.e. at t=TUH.[39] In equation (34), the superscripts H and L refer to low and high damage sub phases, while the superscripts U refers to the business-as-usual phase. The RHS of equation (34) contains the jump in the shadow price of cumulative emissions. It measures the change in the welfare costs associated with emissions per unit of capital before and after the jump. The LHS of

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[36] Although further upward steps in the damage rate are to be expected beyond the climate catastrophe tipping point (i.e. the *TPIRCC*), it is not necessary to specify them—since the model is deterministic and goal of "social planning" is to avoid crossing that threshold. It is important, therefore, that the calibration of the model assures that the latter of these tipping points lies beyond the close of the BAU phase.

[37] The intricacies of the foregoing dynamics are among the reasons why we have not followed the approach taken in by de Bruin, Dellink and Tol (2008) in AD-DICE, in order to explicitly consider the option of defensive expenditures for curtailing anticipated damages to productive capacity resulting from global warming. Because technical changes are assumed to be disembodied in the DICE model, mitigation of C02 induced by rising carbon taxes) does not require specific capital formation to achieve low or zero-emissions production capacity. Consequently, in AD-DICE mitigation does not compete directly with concurrent adaptation expenditures for gross investment allocations. Although the two policies are substitutes when their inter-temporal effects are considered, because effective early mitigation would check the pace of warming and reduce the future need for defensive adaptations, this relationship is not symmetric: early reductions in damage to capacity would then to increase output and accompanying CO_2, but while that would call for more vigorous mitigation efforts the incrementally protect production capacity would not be needed to implement disembodied lower emissions technology. DICE's assumptions regarding technical change thus render the assessment of the interactions between those two options both simpler to model and transparent to assess than would be the case in the present (but in our view more empirically relevant) framework of analysis.

[38] Obviously, the arrival time of the first high damage sub-phase could also be within the joint production phase. But at this stage we are mainly interested in reporting on the principle involved, which would be the same in whichever phase the damages threshold would be situated.

[39] Note that TUH also denotes the end of the low damage sub-phase of the BAU phase. Note therefore that in this case, TUH comes one instant before the value of TUH that represents the beginning of the high damage sub-phase of the BAU phase.

equation (34) measures the welfare cost associated with the extra decay per unit of capital. Equation (34) implies the equality between the welfare cost of using a unit of capital before and after the arrival of the high damage sub-phase.

We conclude that the jump in the co-state must be such that the welfare cost of using a unit of capital remains unchanged. Since the depreciation costs are higher after the jump, the emission costs must be lower, implying a drop (in absolute terms) in the shadow price of emissions. The latter is consistent with the observation that a faster rate of decrease of the carbon-based capital stock will lead to a reduction in the rate at which CO2 emissions are accumulated, other things remaining the same.

4. Model Calibration, Solution Results and Parameter Sensitivity Analysis

4.1 Calibrating the Basic Model and its extended versions

In order to show how the various models work, the parameters of the model need to be calibrated or fixed *a priori*. To this end we have made use of the Nordaus RICE 2010 data,[40] as well as some direct assumptions necessitated by the structural difference the latter model and our DIRAM framework. Nordhaus (2010) updates the calibration of the essential neoclassical growth model setting of the integrated DICE and RICE policy assessment models that have a single, prolonged phase, whereas we employ an extension of the more primitive AK-setting. With multiple phases. This implies that it is not possible simply to import the calibration data for DICE and RICE into our model

Therefore, in order to reproduce global growth rates and saving rates that have about the right size in relation to the "bench-mark" values provided by Nordhaus (2010), we have started with the assumption that the appropriate capital-output ratio for the BAM model in the BAU phase is equal to 4, a value that is well above the capital-out ratio in the multifactor global production system specified in DICE (and RICE). Implicitly our present notion of "capital" must be a more comprehensive one, inasmuch as in the absence of explicit specification of labor as a factor of production, its magnitude must allow for the (proportionality) between conventionally defined tangible capital and all human capital inputs. We have also made the assumption that depreciation costs as a fraction of output is 15%. This also is relatively high value, but actually not so unrealistic, given the much broader capital concept.

Using Nordhaus' data on TFP growth as well as output per capita and population growth, we arrive at an implied growth rate of output (and of capital in an AK-setting) equal to 0.03436. From equation (27) we can derive the steady state growth rate in an AK-setting, and find that on the premise that total output and (carbon-based) capital stock are growing on a steady state path in the BAU phase, at an annual rate equal to 0.03436 (sic!), the following condition must hold:

$$\hat{Y} = \hat{K} = \frac{A - \delta_A - \rho}{\theta} = 0.03436 = \frac{0.25*(1 - \frac{\delta_A}{A}) - \rho}{\theta} = \frac{0.25*(1 - 0.15) - \rho}{\theta} \quad . \tag{35}$$

In equation (35), we have used the assumption that depreciation as a fraction of output is 15%, implying that $\frac{\delta \cdot K}{A \cdot K} = \frac{\delta}{A} = 0.15$. The latter implies that $\delta = 0.15 \cdot A = 0.15 \cdot 0.25 = 0.0375$. This value for the depreciation rate turns out to be much lower than the 0.10 rate used by Nordhaus. Equation (35) implies combinations of θ and ρ given by:

$$\theta = 6.185 - 29.104 * \rho \qquad . \tag{36}$$

Importing the value $\rho = 0.015$ used by Nordhaus, we find that $\theta = 5.748$. This implies a much lower intertemporal elasticity of substitution than the parameter value of $\theta = 1.5$ that Nordhaus has used. In both cases, however, $\theta > 1$. The higher value, however, implies that the function conventionally interpreted as expressing an index of utility or "felicity," namely, $f = C^{1-\theta}/(1-\theta)$, must be negative, although it is increasing in the level of per capita consumption. Marginal felicity would still be positive and decreasing in consumption. Since social welfare is the integral over (the present value) of felicity, the corresponding welfare index also is negative. But the scaling of these indexes is entirely arbitrary, and it is a permissible and simple operation to remove this unaccustomed and disturbing negativity of "welfare" by renormalization of the index of felicity: adding to it, at each moment of time, a positive term equal to the (negative) value of the index of the original index at time zero. This forces an upward shift of the felicity function (and along with it, the corresponding welfare index) into the positive quadrant.

Since θ is relatively large, felicity will be relatively small, as is also the case for the present value of welfare. The numerical values of the co-states consequently will be small as well, since they represent the change in welfare due to a 1 unit change in the corresponding states. For example, since Y_0=68.95 trillion dollars of 2005,[41] our assumptions imply that K_0=275.8, but also that C_0=(1-s).Y_0=49.16. Hence, felicity at t=0 is equal to $F_0 = \frac{C_0^{1-\theta}}{1-\theta} = -1.958 * 10^{-9}$. Therefore we have added a multiplicative scaling factor equal to 10^9 to the felicity function as well, resulting in values for states and co-states that are not many orders of magnitude apart.

Even though our implied inter-temporal elasticity of substitution is rather low, in combination with the other parameter values, plausible values for both the growth rate, and the saving rate can be generated. The implied value of the saving rate (s) is given by:

$$s = \frac{(\dot{K} + \delta \cdot K)}{Y} = \frac{\hat{K}}{A} + \frac{\delta}{A} = 0.287. \tag{37}$$

With respect to cumulative CO2 emissions and the location of the climate tipping point in terms of the cumulative emissions generated by our model, we have used the following procedure. From the Nordhaus data, we can infer that the ratio between the concentration of CO2 in the atmosphere measured in ppm and the concentration of carbon in the atmosphere measured in GTCs is given by:

$$ppm = 0.4695 * GTC \quad => \quad \Delta ppm = 0.4695 * \Delta GTC \quad , \tag{38}$$

where Δ refers to the first difference operator. The present concentration of CO2 in the atmosphere amounts to 390 ppmv.[42] The preindustrial concentration of CO2 is 280 ppmv. The

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[41] t=0 refers to 2010. The data for 2010 are obtained by means of geometrical interpolation between the Nordhaus data for 2005 and 2015.

[42] See the U.S. Department of Commerce National Oceanic & Atmospheric Administration (NOAA) data (at http://www.esrl.noaa.gov/gmd/ccgg/trends/) for trends in monthly mean CO2 concentrations levels measured on Mauna Loa, Hawaii. For the month of October, the mean stood at 388.92ppmv in 2011 and at 391.01 in 2012, giving the annual midpoint of 390 ppm. It should be noted that the trend of these monthly mean observations from the Scripps Institution of Oceanography has been exponentially upwards throughout the period from the late1950s to the present. Over the 32 years 1980-2012 the average annual rate of increase was 0.00476, and projecting this from the 390 ppm annual 2011-12 level, the concentration level will have reached 398 ppm by 2016. Yet , a NOAA news release on 31 May 2012 announced that at Barrow, Alaska – its only remote northern site with continual atmospheric CO2 monitoring -- the concentration level in the spring touched the 400 ppm mark for the first time, in concert with

climate tipping point is generally associated with a temperature rise of 2° Kelvin above preindustrial levels. The equation describing the relation between temperature rises and a CO2 concentrations in ppm relative to some base level is given by:[43]

$$ppmv = ppmv_0 * e^{\Delta T/4.28} \qquad . \tag{39}$$

In equation (39), ΔT represents the temperature rise relative to a baseline temperature at a baseline concentration of CO2 given by ppm_0. It follows that the tipping point at 2° Kelvin relative to preindustrial levels is given by $280\,ppmv * e^{\frac{2}{4.28}} = 446.8\,ppmv$. For a 3° Kelvin temperature rise, the corresponding concentration of CO2 would be 564.4 ppm. The current rate at which the CO2 concentration is rising equals about 2 ppmv/year.[44] Assuming that this rate of increase will be following the rate of growth of output, we find that the room to emit provided by the 2° Kelvin tipping point is equal to 446.8-390=56.8, implying that there can be a further 56.8/2=28.4 '2010 size batches' of gross emissions until the 2° Kelvin tipping point will be reached. According to Nordhaus, the gross emission rate in 2010 is about 10.63 GTC/year. Therefore, 28.4 batches of emissions represent 28.4 x 10.63=301.9 GTC of cumulative emissions in total. A fraction (1-q) of these emissions will be absorbed by the Earth's oceans and forest cover, while the remainder (q) will end up adding to the concentration level in the atmosphere. Using equation (38), the net emissions associated with a rise of 56.8 ppm is given by:

$$\Delta ppm = 0.4695 * \Delta GTC * q => q = \frac{56.8}{0.4695 * 301.9} = 0.4 \quad . \tag{40}$$

It follows that if we rescale the initial level of cumulative emissions to zero, the sum of net emissions until the 2° Kelvin tipping point would be reached is given by $0.4 * 301.9 \approx 120$ GTC. Similarly, the cumulative net emissions for temperature gains of 2.5° and 3° Kelvin would be about 240 GTC and 370 GTC, respectively.

For the magnitude of real gross global product in 2010 we obtain the value 68.95 in trillions of constant 2005 dollars, from the Nordhaus' (2010) DICE calibration data. Since the corresponding initial capital stock value is taken to be 4 times that, the flow of net emissions per unit of carbon-using capital (in GTC per trillion dollar) is found from:

$$\epsilon_A = 0.4 * \frac{10.63}{4 \cdot 68.95} = 0.0154 \qquad . \tag{41}$$

Additional a priori parameter values and adjustments:

For the carbon-free technology we have made the assumption that depreciation is equal to that of the carbon-based technology. In addition, for the Basic Model the average (and marginal) productivity of carbon-free capital is set at $B = 0.12$. Further, in specifying the R&D impact function we have made the following parameter assumptions: $\beta = 0.5$, $\zeta = 0.1$, and $\bar{B} = 0.2$.

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the same "milestone" readings reported by remote northern monitoring sites in Canada, Finland, Norway and an island in the north Pacific. This has supported the consensus view among NOAA's climatologists and their international colleagues that by 2016 mean global CO2 concentrations will have risen to 400 ppm (cf. http://researchmatters.noaa.gov/news/Pages/arcticCO2.aspx.).

[43] For further details, see http://en.wikipedia.org/wiki/Radiative_forcing.

[44] See the NOAA (http://www.esrl.noaa.gov/gmd/ccgg/trends/) data for the annual increases in the CO2 concentrations levels measured on Mauna Loa, Hawaii. Calculated for the years 2002-2012 (from the time-series available at ftp://ftp.cmdl.noaa.gov/ccg/co2/trends/co2_gr_mlo.txt), the average annual gain was 2.06 ppm.

Lastly, we have adjusted the climate tipping point in order to have a business-as-usual phase longer than just 5 or 10 years for the 2° Kelvin temperature rise. In fact, we have used a value of 325 GTC net which is consistent with a temperature rise of about 2.75° Kelvin. So that the expected higher rate of losses of capital services (due to warming-driven extreme weather events) will emerge within the endogenously lengthened business-as-usual phase, we have set that threshold level at 87 GTC (net) cumulative CO2.

4.2 BAM solution results

Preliminary parameter sensitivity analyses performed using the models show model reactions that are familiar from growth theory. Changes in the rate of discount or in the intertemporal elasticity of substitution all have the expected impact. This holds for the productivity parameters as well. When cumulative emission thresholds are tightened, the shadow price of CO2 emissions rises (in absolute terms). When productivity parameters increase, so do the corresponding co-states of the associated state variables.

Rather interestingly, the linking of various sequential phases results in anticipatory adjustments that introduce transitional dynamics which are missing in an ordinary single-phase AK endogenous growth setting. The results obtained for the Basic Model (BAM) with the parameter set and the initial values described at the end of the preceding sub-section (4.1) are displayed in Figures 2.1 and 2.2.
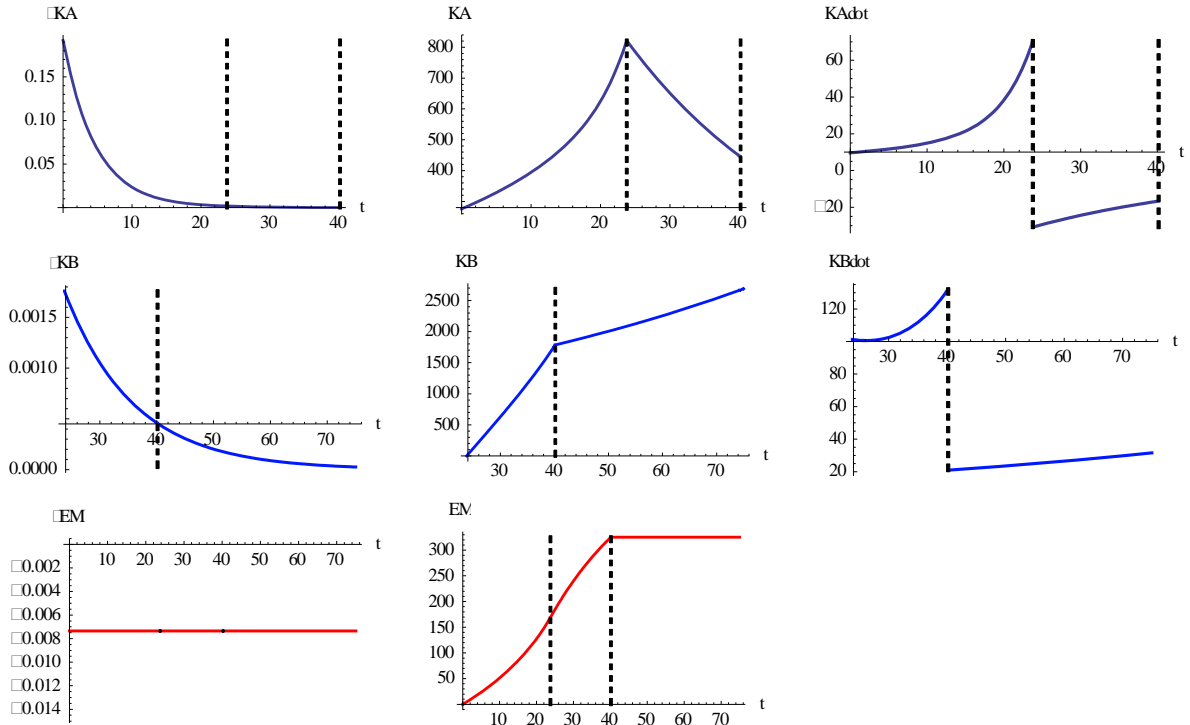
Figure 2.1 BAM baseline results for K_A, K_B and E

The first row of plots in Figure 2.1 displays the outcomes with respect to carbon-based capital K_A. The vertical dotted lines mark the arrival times of the joint production phase and the carbon-free phase. They are situated at TJ=23.78 and TF=40.17, implying that the BAU phase takes slightly less than 24 years, while the JPR phase is slightly longer than 16 years. The first graph in this row shows the shadow price of carbon-based capital decreasing steadily throughout both phases. The next graph shows the buildup of carbon-based capacity during the

opening, BAU phase and the subsequent run down of that capacity during the JPR phase. These events are mirrored in the last graph in the row, which plots the instantaneous rate of change in the carbon-based capital stock. We see that net investment in carbon-based capital accelerates towards the end of the business-as-usual phase, and turns negative during the joint production phase -- although becoming less negative over time as the absolute amount of capital lost due to technical depreciation of given size of stock K_A is becoming smaller.

The second row of graphs in Figure 2.1 shows the corresponding events for carbon-free capital. Since the accumulation of carbon-free capital begins at the start of the joint production phase, there is now just one dotted vertical that marks the arrival of the carbon-free phase. It should be noted that net investment in carbon-free capacity is rapidly increasing during the joint production phase -- in anticipation of the drop in capacity that will occur at the time of arrival of the carbon-free phase, when carbon-based capital will be discarded. During the carbon-free phase, net investment in the carbon-free capital stock is much lower than during the joint production phase.

In the third row, the left-most graph depicts the exponential rise of the atmospheric concentration of CO2 (accumulating from the past flow of emissions) during the business-as-usual phase. These continue to rise but at slowing growth rates over the course of the joint production phase, eventually stabilizing at the threshold level when the economy has entered the carbon-free phase of production. The corresponding shadow price of cumulative emissions is negative, since it reflects the social cost of adding an incremental unit to the stock of atmospheric CO2. The constancy of this negative shadow price throughout the transition to the carbon-free production regime is a striking departure from the familiar optimal solution results obtained by Nordhaus (1994, 2000, 2007) with the DICE model and its close variants, as well as by other contributions to the IAMS climate policy assessment literature that find a time-path for the "social cost of carbon" --i.e., that of the current flow of CO2 emissions -- that gradually "ramps up" *pari passus* with the rising cumulative stock of greenhouse gas in the atmosphere.[45]

The explanation for this apparent "anomaly" is simply that, unlike most of the IAM studies, in our models the effects of cumulative emissions are not represented as "damages" that directly and continuously enter the social welfare function. In BAM, the current level of cumulative emissions, taken in combination with a pre-specified level of the climate-tipping point (marking the onset of the climate catastrophe that is to be averted), set the remaining permissible volume of cumulative CO2 emissions. So long as that volume is positive (actual cumulative emissions being below the critical threshold), the complementary slackness theorem tells us that the product of the Lagrange multiplier μ and its corresponding inequality constraint set by the threshold level should be zero at all times. Hence, so long as there is "room for maneuver" μ should be zero. In optimum control terms this translates into the fact that the derivative of the Hamiltonian with respect to cumulative emissions equals $-\mu = 0$, making the time-derivative of the corresponding co-state equal to zero too, thereby implying that the co-

state itself (the shadow price) should be constant over time.[46] As will be seen (below, in sect. 4.2), the foregoing results are found also in the extended versions of BAM that recognizes that rising atmospheric CO2 concentration levels will drive "damages" from the increasing frequency of severe weather events and climatic shifts causing coastal flooding and interior droughts, all of which translate into effective losses of productive capacity.[47]

Figure 2.2 presents the corresponding BAM paths found for total output, consumption, along with their respective growth rates, and the indexes of "felicity" (F) and social welfare (W). Note that the the rescaling of the "felicity function" has yielded positive values for both that index and W, aswill be seen from the bottom-most row of this Figure.
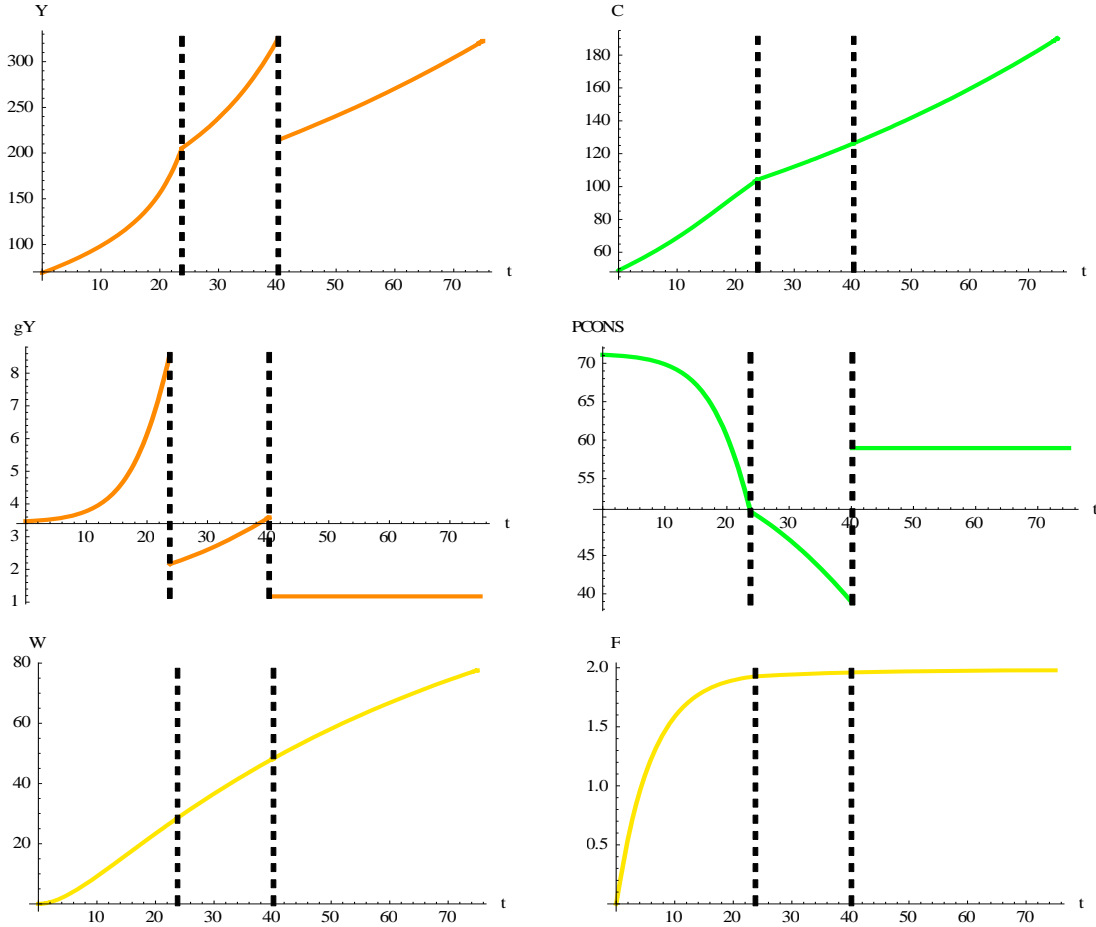
Figure 2.2 BAM baseline results for Y, C, F and W

The growth rate of output shows some anticipatory reactions to the changes that the arrival of a new phase will bring. For example, during the joint production phase, the average productivity of capital must fall, as the amount of relatively productive carbon-based capacity decreases, and the amount of relatively unproductive carbon-free capacity increases. This holds

[46] Since, by construction, the threshold is not binding during the BAU and the JPR phase, the emission constraint hasn't been explicitly introduced in the Hamiltonians pertaining to both phases. If they would have been, complementary slackness would have 'neutralised' them for the reasons given here.

[47] The damages, however, are not modeled as losses of "environmental amenities" which would impinge directly upon "fecility" and therefore be treated as directly damaging social welfare. Instead, they impinge upon welfare only indirectly through the optimizing resource allocation adjustments that have to be made to compensate for the anticipated losses of the capital stock by lowering present and future levels of consumption.

a fortiori for the arrival of the carbon-free phase: when the remaining carbon-based capacity is discarded, aggregate capital productivity suddenly drops to the level associated with carbon-free capacity. In order to mitigate the effects on the consumption path of the corresponding drop in output, the buildup of carbon-free capacity during the joint production phase should be speeded up towards the end of the production phase.

A similar pattern can be observed (in Figure 2.1) for the buildup of carbon-based capacity during the business-as-usual phase, as an increase of the carbon-using capital stock also allows a relatively high rate of investment in carbon-free capacity during the next phase. This nicely illustrates a general implication of technical innovations that must be physically embodied in production facilities, one that often is overlooked in discussions of policies to mitigate CO2 emissions: because capital is a produced means of production, a fast buildup of carbon-free capacity will require a large pre-existing stock of carbon-using capital – at least during its initial stage.

In Figure 2.3 we report the consequences for the shadow prices of cumulative emissions and of carbon-based capital of altering the location of the "climate catastrophe tipping point," in the BAM setting. This sensitivity experiment varies the critical CO2 concentration level over the range 125-375 ppmv, which is consistent with temperature gains of 2-3 degrees Kelvin above preindustrial levels. From the lower left plot is seen that when is the *TPIRCC* is set at the bottom end of its range, i.e., at Emax =125-130 corresponding to an atmospheric CO2 concentration level (405-410 ppmv) not far so above the 390 ppmv mark reached by the average record reported for 2011-12 by the observatories on Mauna Loa (cf. footnote 39), the remaining "emissions budget" is so small that there are less than 10 years left to accomplish whatever has to be done before end of the business-as-usual phase. Thereafter, the duration of the BAU phase increases almost linearly and less than proportionately in relation to the further relaxation of the constraint set by the "permitted budget" for total cumulative emissions, but each "relaxing
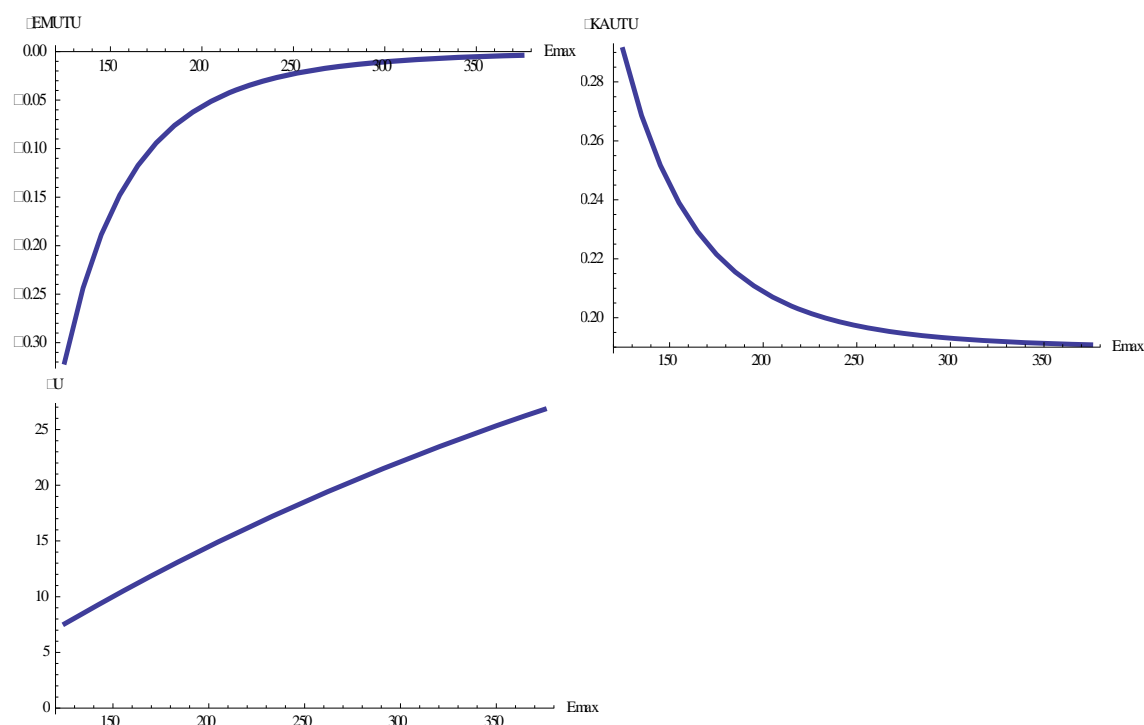
Figure 2.3 Sensitivity results BAM

step" will be raising the mean global temperature that prevails at the beginning of the switch to carbon-free capital formation, and by the time another 9 years have been allowed for the (optimized) BAU duration, the temperature gain in this simple illustrative model would have will have exceed the "dangerous" 2ºC.

Since the carbon-free end-phase has a constant steady state growth rate, we can limit ourselves to looking only 75 years into the future, and thus showing just the first part of the corresponding time paths following the successful completion of the climate stabilizing transition. Figures 3.1A and 3.1.B show investment for values of the emissions threshold that are varied uniformly over this range. Low threshold values are associated with the low frequency part of the rainbow color spectrum, and higher values of the threshold correspond with correspondingly higher color frequencies, ending with the violet part of the color spectrum for the highest threshold value within the range. Hence, the red lines correspond with a tight emission constraint and the violet lines are associated with the loosest emission constraint.
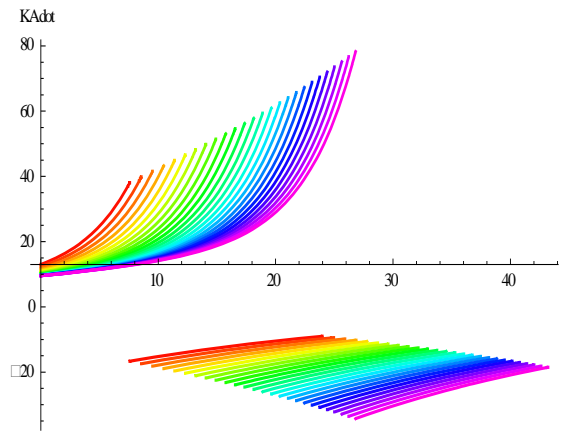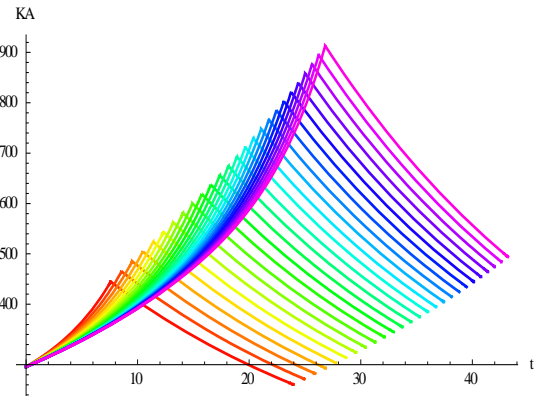
Figure 3.1.A $\dot{K}_A$: Emax=125-375   Figure 3.1.B K_A : Emax=125-375

We can observe that as the threshold becomes tighter the arrival of the carbon-free phase is speeded up. Figure 3.1A shows two blocks of lines. The leftmost block is associated with the business-as-usual phase. The rightmost block is associated with the joint production phase. Note that the endpoint of a particular time path in the leftmost block coincides in time, but not necessarily in value, with the initial point of that same time path in the rightmost block. In Figure 3.1.B these points not only share the same time coordinate, but also the same value of the capital stock. This is because states and co-states cannot jump, whereas time derivatives can.

One sees immediately from Figure 3.1.B that when the emissions constraint is loosened, the business-as-usual phase lengthens and, for a given duration of the joint production phase, that implies and equal-lenght postponement of the arrival time of the carbon-free phase. But the effect of relaxing that constraint is that the phase of joint production (and investment in carbon-free capacity) also becomes longer. There is a striking difference, however, between the way that varying Emax affects the net investment patterns for carbon-based capital, and for carbon-free capital. As the emission constraint is loosened, the whole of the net investment curve for carbon-free capital is shifted upwards throughout the entire joint production period. In the case of carbon-based capital, however, a loosening of the emission constraint implies both a lengthening of the business-as-usual phase and a downward shift of the net investment time path. At the end of the business-as-usual phase, however, the downward shift is more than compensated by the rise in net investment taking place over a longer stretch of time. The counterpart of this sequence of events is shown in figure 3.1.G which shows the time paths of

consumption. There we see that the BAU phase allows higher levels of consumption in the beginning, and correspondingly lower levels of net investment in carbon-free capacity.
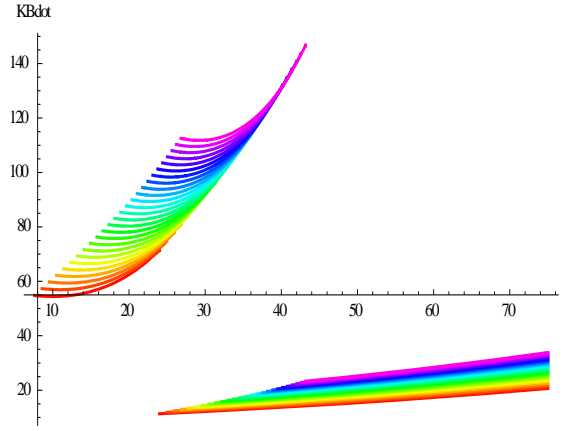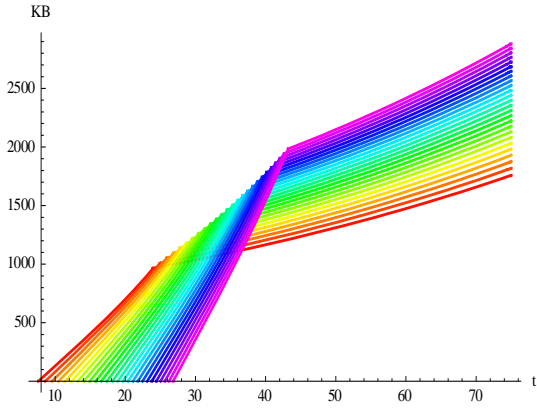

Figure 3.1.C $\dot{K}_B$: Emax=125-375

Figure 3.1.D K_B : Emax=125-375

The implications for the level of the carbon-free capital stock are shown in Figure 3.1.D. With a tighter emission constraint the joint production phase comes earlier, while the capital stock reaches a lower level at t=75 and hence will be lower also at $t = \infty$. Once again, this is a consequence of the fact that capital is a produced means of production: if the carbon-based capital stock is limited in size by the existence of more binding cumulative CO2 emission constraints, then the implication is that the carbon-free capital stocks will be more limited in size as well.
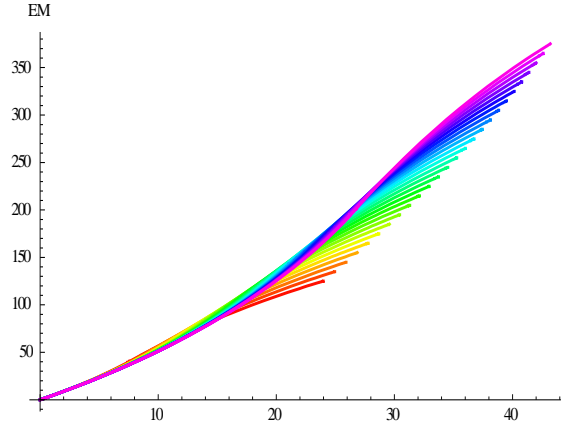
Figure 3.1.E Cumulative emissions, E : Emax=125-375

Obviously, more limiting emissions show up directly in the time paths of cumulative emissions, both in terms of the value of the endpoints, but also in terms of the associated time coordinate, as can be seen from Figure 3.1.E. Events for output are more interesting, since output in the most constrained case reaches higher levels at the end of the BAU phase, but then loses out to the less constrained cases, as shown in Figure 3.1.F (below). That twist in the relative levels of output, however, is not mimicked by the time paths of consumption which are seen from Figure 3.1.G to growth in parallel. Throughout the entire duration of the transition it is the less tightly constrained solutions that yield the persisting higher levels of per capita consumption.
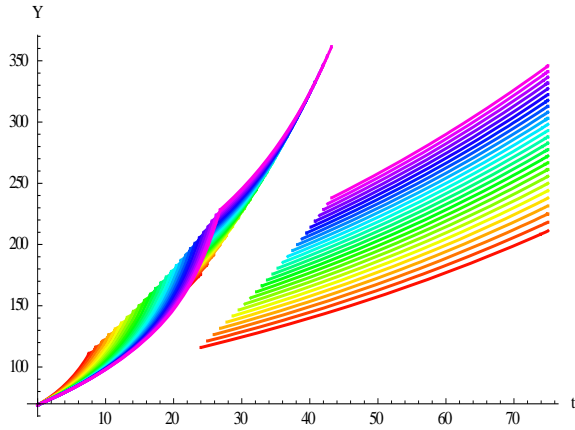
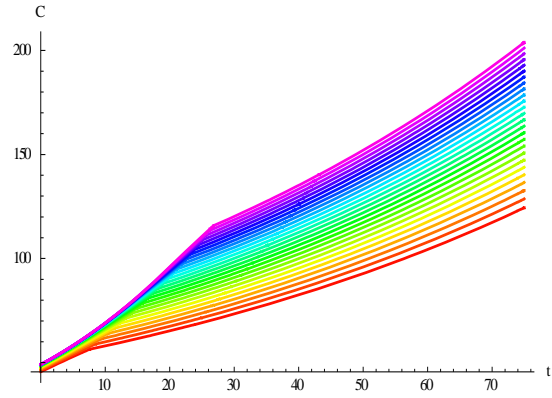Figure 3.1.F Output: Emax=125-375

Figure 3.1.G Consumption: Emax=125-375

The twist in the relative levels of output, just mentioned, is reflected by the pattern of growth rates depicted in Figure 3.1.H. The values of the growth rate of output at the end of the business-as-usual phase are the same for all threshold values within the range. However, the periods of time during which these growth rates affect the level of output are very different, so much so that the positive effect on the level of output (see Figure 3.1.F) of an extension of the BAU phase with the relaxation of the Emax constraint outweighs the intial negative effect of that change on the rate of growth of output. The fact that the model converges to the same steady-state, albeit at different points in time, is reflected by the horizontal line in the rightmost part of Figure 3.1.H
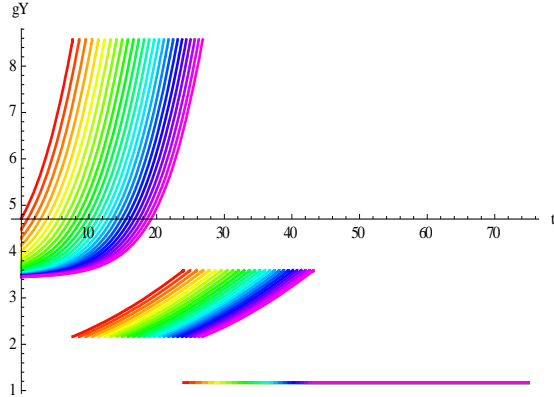
Figure 3.1.H Growth rate $\dot{Y}/Y$: Emax=125-375
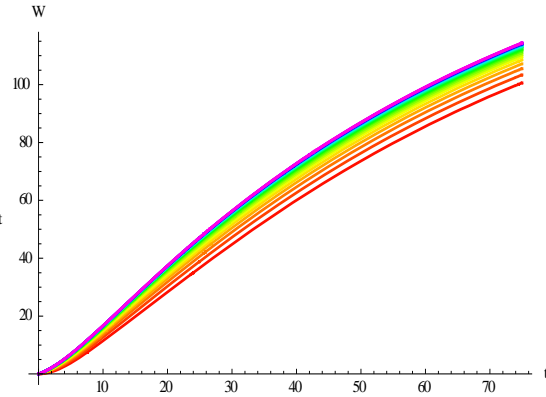
Figure 3.1.I Welfare W : Emax=125-375

The effect on welfare of less stringent constraints on cumulative CO2 emissions is positive, as one would expect, as can be seen in Figure 3.1.I. In addition, the time paths become closer together in the vertical direction with the relaxation of that constraint. The implication of this is that both the welfare effects associated with the trade-offs between consumption and investment during the earlier BAU phase, and those welfare effects in the joint production phase, become stronger when tighter constraints on emissions shorten the BAU phase.

4.3 Solution results for BAM+R&D

The base line results for BAM+R&D are presented in Figure 4.1. The length of the BAU phase and the JPR phase are now 27.36 and 13.45 years. Hence, the arrival date of the carbon-free phase has been ever so slightly postponed relative to BAM (TF=40.81 for BAM+R&D and

TF=40.17 for BAM), but the main difference is in the relative lengths of the BAU and JPR phases. With endogenous R&D, the BAU phase is lengthened from a value of 23.8 to 27.4 years, whereas the length of the JPR phase is reduced from 16.4 in BAM to 13.4 in BAM+R&D. This is an illustration of the fact that accumulation of physical carbon-free capital with low productivity for longer periods of time (as in BAM) is a substitute for the accumulation of high productivity carbon-free capital for shorter periods of time (as in BAM+R&D) *plus* the accumulation of productivity change through R&D prior to the JPR and CFR phases.

The big difference between the results found (below) in Figure 4.1 and those shown for BAM in Figure 2.1 is the presence of the third row of plots that is now associated with the capital productivity B of the carbon-free technology: the shadow price of capital productivity rises during the business-as-usual phase, and then falls.
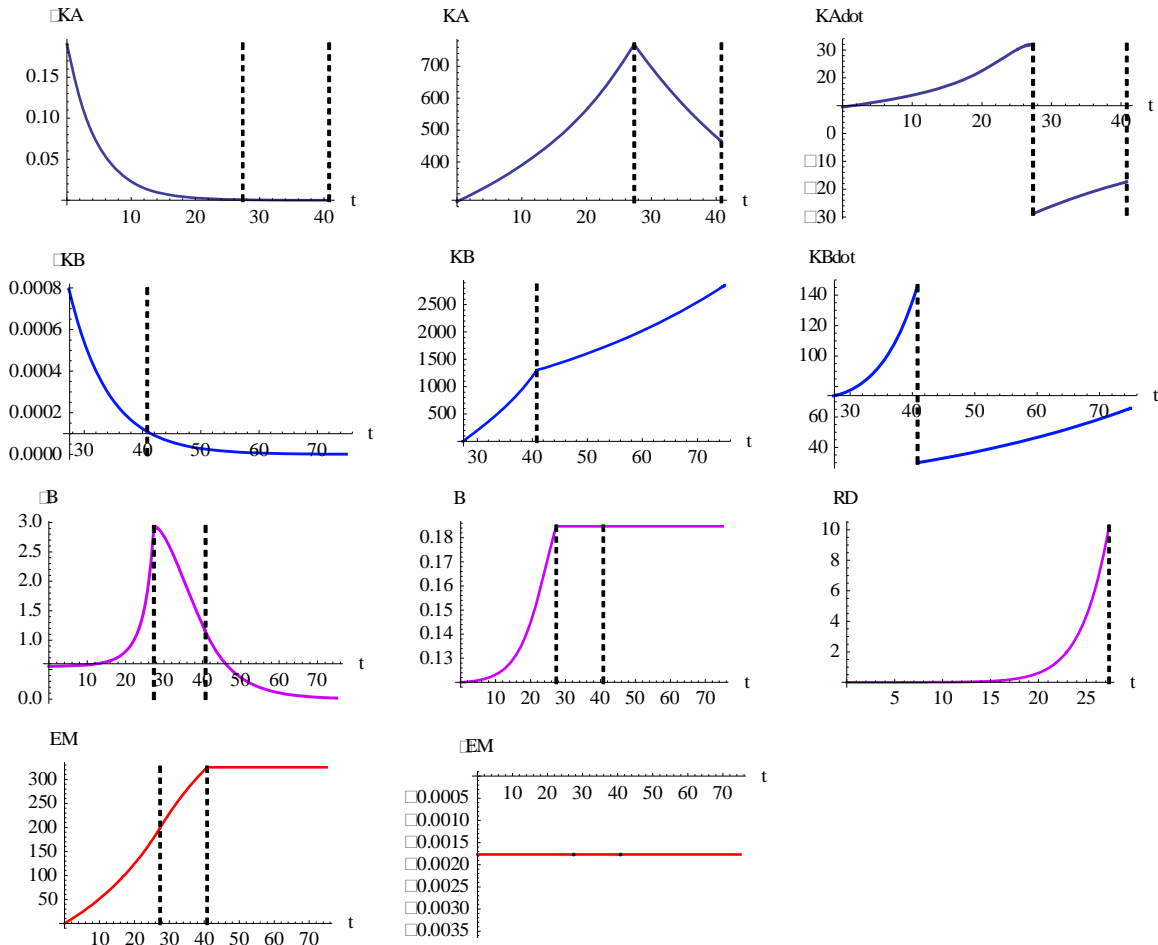
Figure 4.1 BAM+R&D baseline results for K_A, K_B and E

The reason why the shadow price rises at first is that during the business-as-usual phase the capacity to produce carbon-free capital that will embody the new value of the capital productivity is rising. This represents an increase of the value of doing R&D. But since the productivity of doing R&D is high for low values of the productivity of capital, the corresponding impact on R&D levels is positive but limited, as long as B is relatively low. As *B* approaches its asymptotic value, however, the level of R&D activity increases exponentially, but this happens at a relatively late stage in the business-as-usual phase. At the close of that phase R&D activity

ceases and B remains at a constant level (below its asymptotic value) while the shadow price of B will be falling from then onwards.
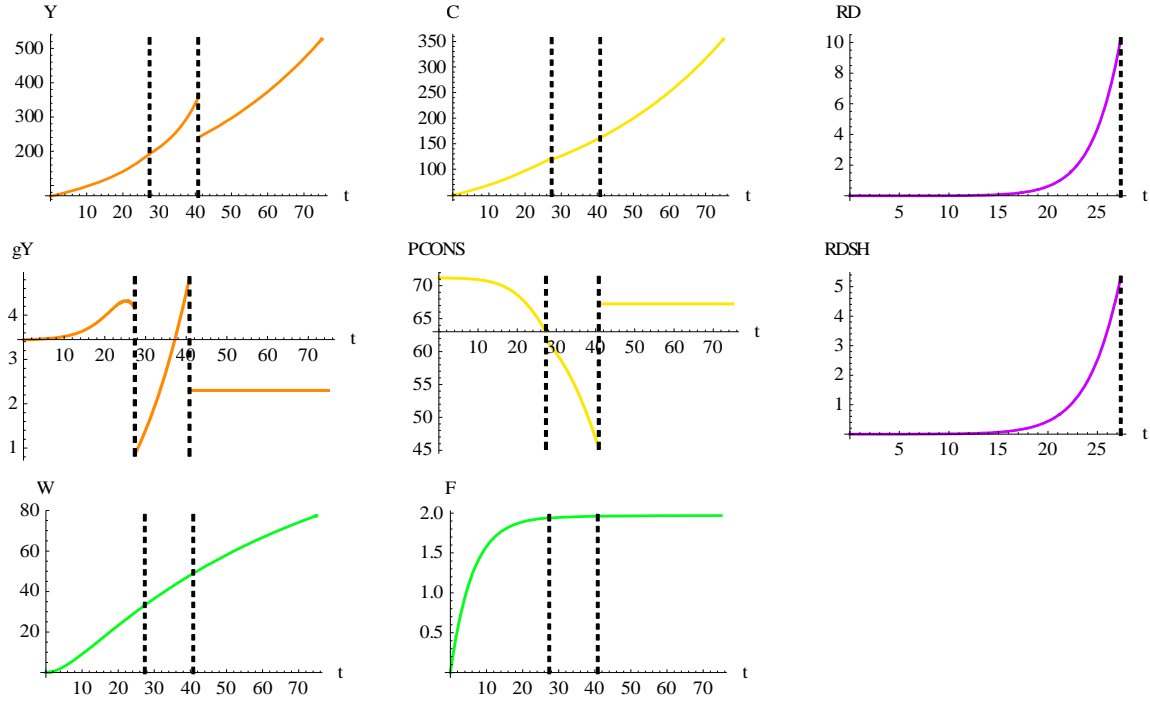

Figure 4.2 BAM+R&D baseline results for Y, C, R&D, F and W

Figure 4.2 shows the absolute R&D expenditures (RD) as well as the percentage share of R&D expenditures in total output, from which it can be seen that the exponential rise in R&D expenditures at the end of the BAU phase is reflected in the slowdown of the growth rate of Y at the end of the BAU phase. The retardation in output growth is traceable to the slowing of net investment in the carbon-based capital stock to accommodate the increase in R&D expenditures. During the joint production phase the rate of net investment in carbon-free capacity is significantly higher in the case of BAM+R&D than it was in the Basic model, because the effect of R&D activities during the preceding phase has raised the marginal rate of return on investment in carbon-free capacity, relative to the level at which it was exogenously fixed in the case of BAM.

For BAM+R&D we have run the same sensitivity experiment regarding the location of the cumulative emission threshold as was carried out for the BAM model. Figure 4.3 presents the initial values for the co-states of cumulative emissions ($\lambda EMUTU$), of carbon-based capital ($\lambda KAUTU$)and of the productivity of carbon-free capital($\lambda BUTU$), as well as the length of the BAU phase (ΔU) and the JPR phase (ΔJ).

The plots for the initial values of the co-states of cumulative CO2 emissions, and of carbon-based capital are very similar to the ones we had obtained for BAM, whereas the plots for the initial value of the shadow price of carbon-free capital productivity show that tighter emission constraints tend to raise the value of doing R&D, as one would expect. We also find that the BAU phase has been lengthened by a couple of years, across the board. One of the reasons for this result is that the strongly concave relationship between R&D expenditures in increases in the productivity of capital embodying the carbon-free technology introduces a tendency for R&D investment to be spread out over time. Consequently, the longer the BAU phase, the higher the benefits that can be obtained by spreading out R&D expenditures.
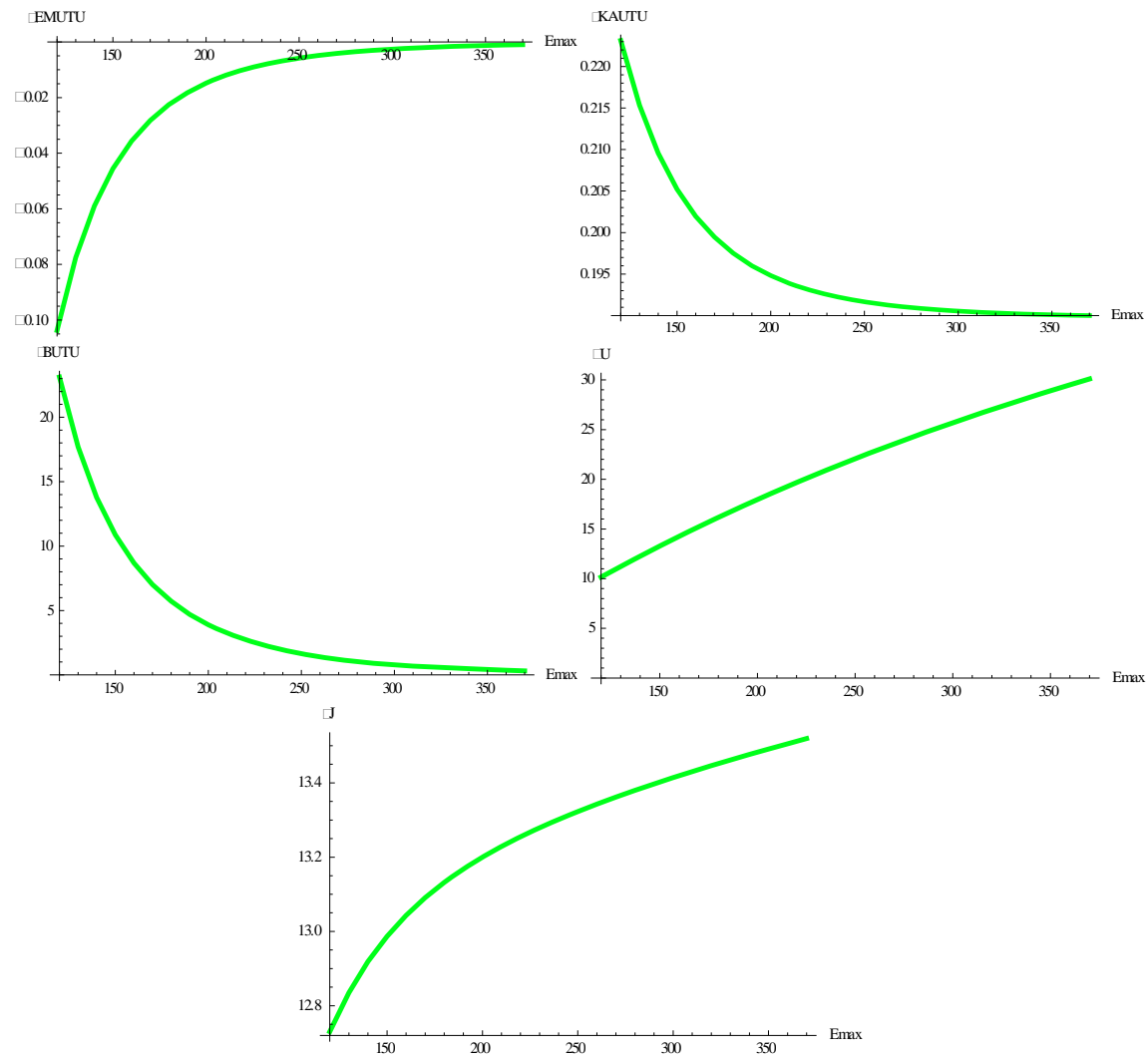
Figure 4.3 Sensitivity results BAM+R&D

In discussing the optimal solution found for the BAM transition path, it was pointed out that the duration of the JPR phase depended only on the technology parameters, which remained fixed in the BAM setting. Endogenous R&D activity, however, causes B to change over time. As a consequence, the effects of varying the *TPIRCC*'s location in the BAM model are quite
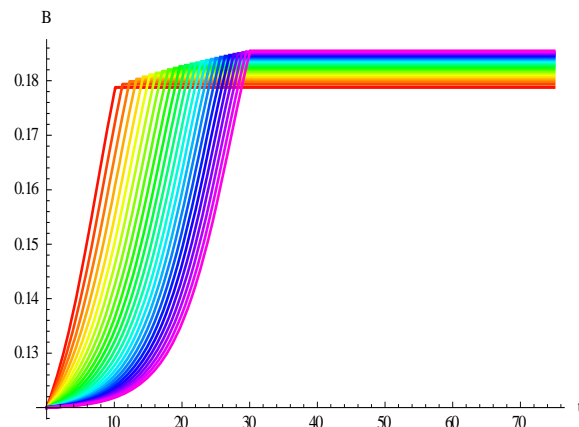

Figure 5.1 Time paths of capital productivity B

different from those found in the BAM case not only in the BAU phase, but in the duration of following joint production phase. Figure 5.1 displays what happens to the development over time of carbon-free capital productivity in this model when the constraint imposed on CO2 emissions by the climate catastrophe tipping point is relaxed. One sees, first, from the spread between the spread between the red and the violet bounds, that an increasingly distant tipping point allows the pace of R&D-driven productivity enhancements (during BAU) ramp up more slowly, stretching out that first phase of the transition, but raising the terminal value of carbon-free capital productivity (B). The latter translates into a lower aggregate capital-output ratio and a faster rate of steady-state growth during the eventual epoch of carbon-free production.

To appreciate the complicated collateral changes in the behavior of tangible investment during the BAU phase, it is helpful to start with a comparison of Figures 4.1.A, B with Figures 2.1.A, B. From this it is seen that the duration of the (BAU) phase--when net investment is building up the carbon-using capital--is noticeably more prolonged in the case in which the R&D option is being exercised (BAM+R&D) than when it not (BAM). Nevertheless, in the BAM model a higher and accelerating growth rate of the carbon-using capital stock results in the BAU phase ending with the stock of that type of capital being substantially bigger than its counterpart in the optimized solution for the BAM+R&D model. With regard to the behavior of tangible net investment, therefore, there is a much less abrupt passage in the latter model between the BAU phase the following (joint production) phase of the transition path.

Looking now at the contrast between the solutions for these two models with regard to the effects of variations in the location of the tipping point (*TPIRCC = E*max), Figures 3.2.A,B shows that in the BAM case the more distantly positioned is the constraint on *E*max, the longer is the "stretch-out" of net investment in carbon-using capital, thus deferring the largest annual changes to the later years of the more prolonged BAU phase. When the option to improve the productivity of carbon-free capital through R&D is being exercised, however, capital formation to increase carbon-based productive capacity continues for a much shorter time. Other comparisons reveal that the level of net investment in carbon-based capacity similarly is lowered more in the case of BAM+R&D than it is in the model where the option to invest in R&D is not present.

On the other hand, the effects of varying Emax on the terminal values of the carbon-based capital stock at the end of the BAU phase are found to be similar in magnitudes, and the time paths of the stock of carbon-based capital remain packed together more tightly in the case of BAM+R&D than they are with BAM alone. For the carbon-free capital stock, however, the rate of net investment under BAM+R&D rises more rapidly over time than under BAM, and the periods during which the buildup of carbon-free capacity is realized, are somewhat shorter.
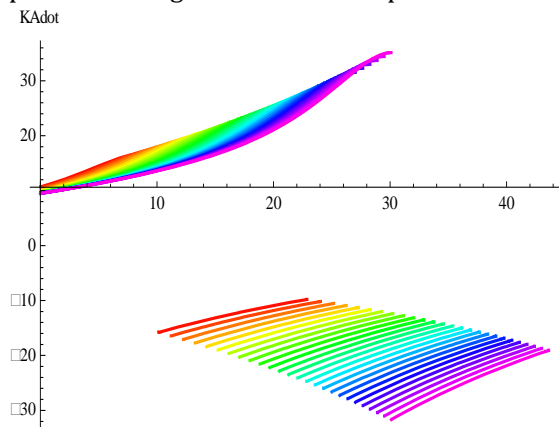

Figure 5.2.A $\dot{K_A}$: Emax=125-375   Figure 5.2.B K_A : Emax=125-375
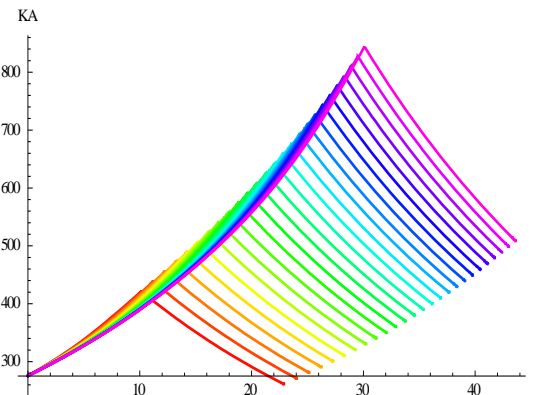
In all cases, however, the carbon-free capital stock at the end of the JPR phase under BAM+R&D exceeds that of the corresponding terminal value under BAM. But, varying the climate catastrophe tipping point generates a spread in terminal values for the carbon-free capital stock at t=75 that is much smaller in the BAM+R&D model than those found when there is no R&D.
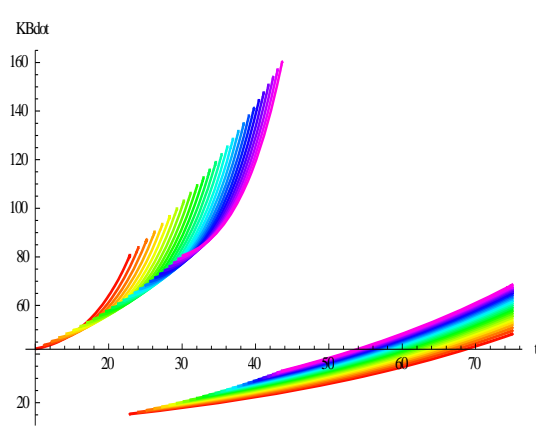

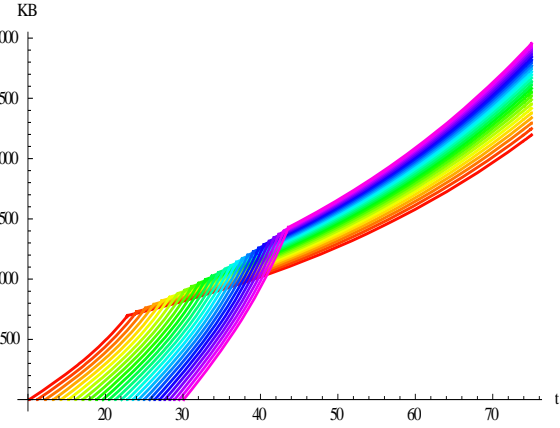Figure 5.2.C capital $\dot{K_B}$: Emax=125-375

Figure 5.2.D K_B : Emax=125-375

With respect to the time path of emissions, Figure 5.2.E exhibits the "shock absorbing" effect of exercising the R&D option, damping the impacts of the tighter constraints imposed by setting the tipping point at lower CO_2 concentration levels. From a comparison with the results for BAM (in Figure 3.2.E) it is evident that the time-paths of cumulative emissions virtually coincident throughout a large part of the somewhat longer BAU phase, and the subsequent "feathering" of the paths is narrower: a range of 40 *ppmv*, compared with 70 *ppmv* in BAM.
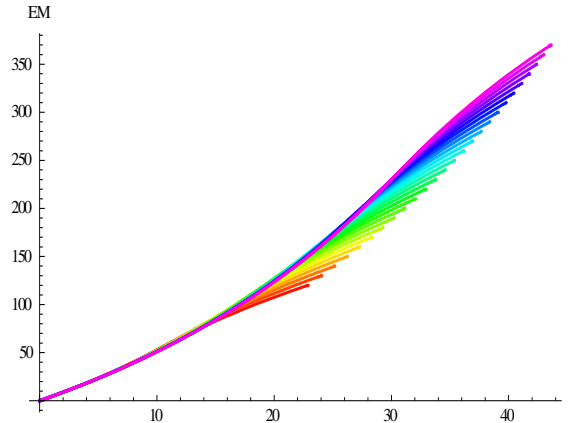

Figure 5.2.E Cumulative emissions, E : Emax=125-375

The more even development over time under BAM+R&D than under BAM is also reflected in the plots regarding output. The kinky growth patterns observed under BAM are far less pronounced in the case of BAM+R&D (compare. Figures 5.2.F and 3.1.F as well as 5.2.H and 3.1.H), and this holds also for consumption (Figures 5.2.G and 3.1.G).

One difference between BAM+R&D and BAM is very noticeable from the comparison of Figures 5.2.H with 3.1.H. First of all, during the BAU phase under BAM+R&D the average growth rate of output is lower than the average growth rate under BAM. Secondly, under BAM+R&D, the average growth rate slows down at the end of the business-as-usual phase. During the joint

production phase, however, the range of variation of the growth rate of output is much larger under BAM+R&D than under BAM, while, moreover, the steady state growth rate under BAM+R&D is higher than under BAM and slightly rising as emission constraints become less tight. All of this leads to a much more even development over time of consumption, welfare and felicity, as can be seen by comparing Figures 5.2.G and 3.1.G as well as Figures 5.2.I and 3.1.I. These Figures highlight the fact that having the possibility to change productivity through R&D helps to fight the negative effects of tightening emission constraints.
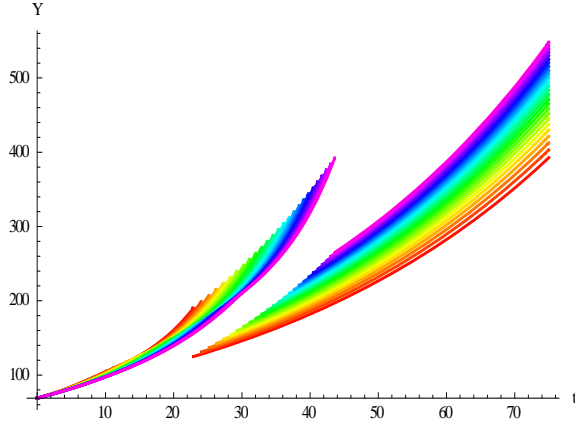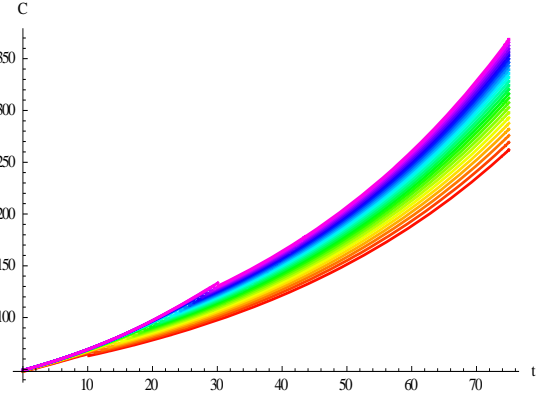

Figure 5.2.F Output : Emax=125-375
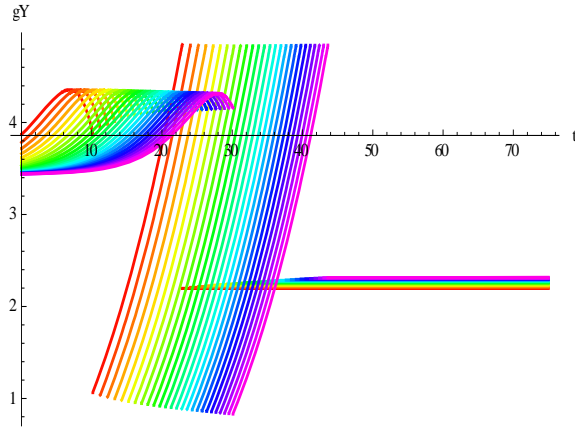

Figure 5.2.GConsumption : Emax=125-375

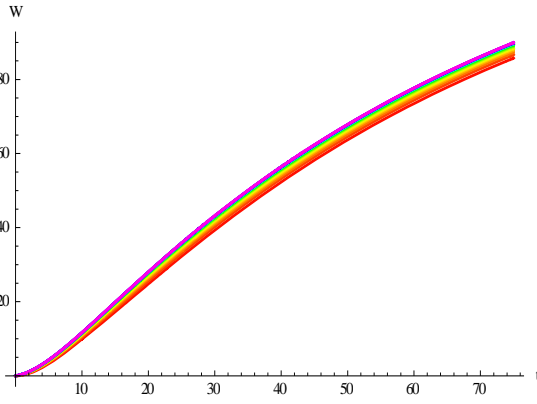Figure 5.2.H Growth rate $\dot{Y}/Y$: Emax=125-375

Figure 5.2.I Welfare W : Emax=125-375

4.4 Solutions results for BAM+R&D+UCL : Sensitivity Experiments

The results for this experiment have been obtained as follows. We have first made the assumption that the high damage depreciation parameters are exactly the same as the low damage parameters, in which case the model is reduced to the BAM+R&D model, providing a base-line against which to gauge the effects of recognizing higher rates of warming-driven damages. Then we set damage threshold well within the BAU phase of the BAM+R&D model. With the onset threshold situated at 87 GTC net cumulative emissions from current levels, the high-damage threshold as reached after 15 years. To perform the sensitivity analysis that is reported here, the depreciation parameter for carbon-based capacity was varied over the range 0.0375-0.075, i.e. twice the initial value. In other words, the largest increase in the expected rate

of annual gross weather-related damages considered here is 3.75 percent of real gross output, with the 1.875 percent being the mid-point of the parameter variation range.[48]

Because this model is so similar to BAM+R&D, we do not report the outcomes of the climate change threshold here. Rather, we focus on the experiment in which we account for extra damages for carbon-based capital. The damage threshold effectively splits the BAU phase into a low damage first sub-phase and then a high damage sub-phase. The joint production phase and the carbon-free phase now will now also be high damage phases. The length of the several phases With the physical depreciation value of $\delta_A^H = \delta_A^L = 0.0375$ are 15.36 for the low damage sub phase of BAU (i.e. ΔUL), 12.0 for the high damage sub-phase of the BAU phase (i.e. ΔUH), and 12.83 for the JPR phase (i.e. ΔJ).

It is clear from Figure 5.3 that a rise in the decay parameter in the high damage phase will bring about a rise (in absolute terms) in the initial shadow price of emissions. This is reflected also in Figure 5.4, which displays the magnitudes of the effect on the shadow price of cumulative CO2 emissions for the first 15 years of the transition. In Figure 5.4 it also is evident that the absolute elevation of the shadow price of CO_2 emissions is an anticipatory effect of the future rise in the rate of losses from the net stock of capital, and is confined to the low damage sub-phase. From the high damage BAU sub phase onwards, the shadow price remains unaffected by rises in the rate of decay, suggesting an absolute rise of the shadow price as compared to the BAM+R&D baseline results. The reason why this absolute rise occurs is that the net rate of return on carbon-based capital in the low damage sub-phase has risen relative to the high damage sub phase. Consequently, there is more demand for carbon-based capital during the low damage sub-phase, and therefore also more derived demand for the existing room left for cumulative emissions before the climate catastrophe tipping point will be reached.

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[48] While the step-function specification is an arbitrary simplification, the choice of this step-size range is not entirely *ad hoc*. Although the step-function specification is an arbitrary simplification, the choice of this step-size range is not entirely *ad hoc*. One may starting with a continuous specification that makes the proportional (weather-driven)damage rate (Dw/Y) a positive power-function of the gain in global mean temperature (ΔT): $(Dw/Y)_t = \alpha_1(\Delta T)_t + \alpha_2(\Delta T)^{\alpha_3}{}_t$. This is the specification for the gross damage rate used in the AD-DICE model, with the parameter values $\alpha_1 = .0012$, $\alpha_2 = .0023$, $\alpha_3 = 2.32$. (See de Bruin, Kelly and Tol 2009: esp. 68-69 for the calibration proceedure used to find the value of the exponent that was consistent with simulation output from AD-DICE giving the best fit to DICE simulations – the latter model being one that implicitly nets out adaptive benefits against damage curtailment costs, leaving net(residual) damages. Using the standard relationship for the equilibrium gain in temperature corresponding to an increase in atmospheric CO2, denoted here by $(C_t - C_0) = E_t$, the proportinate increase in the concentration level may be expressed as $C_t/C_0 = [(E_t/E_0) + 1]$, allowing us to find:
$\Delta T^* = S[(ln\{E_t/E_0\} + 1)/(ln 2)]$, where the equilibrium climate sensitivity parameter $S = 3$, consistent with the magnitude accepted by Nordhaus (2007,2010) in calibrating DICE. From this we find the following correspondences between temperature gains and expected proportional rates of weather-related losses of carbon-using productive capacity (gross output, in our linear production system): $\delta_A^H = .0012$ at t=0, with $\Delta T = 1°$ Kelvin; .0087 at c. 2010, with $\Delta T = 1.61°$; at .0228 with $\Delta T = 2.53°$; at .00517 with $\Delta T = 3.68°$. The corresponding range of variations in the "high-damage" rate considered in the sensitivity analysis therefore is $E_t = [125, 375]$, or in absolute concentrations (ppmv) levels (Et+280) = [405, 655]; the mid-point of this range corresponds to $\delta_A^H = .00373$. Recall, then, that the *constant* UCL rate specified by our model was set (coincidentally) at 3.75 percent per annum.

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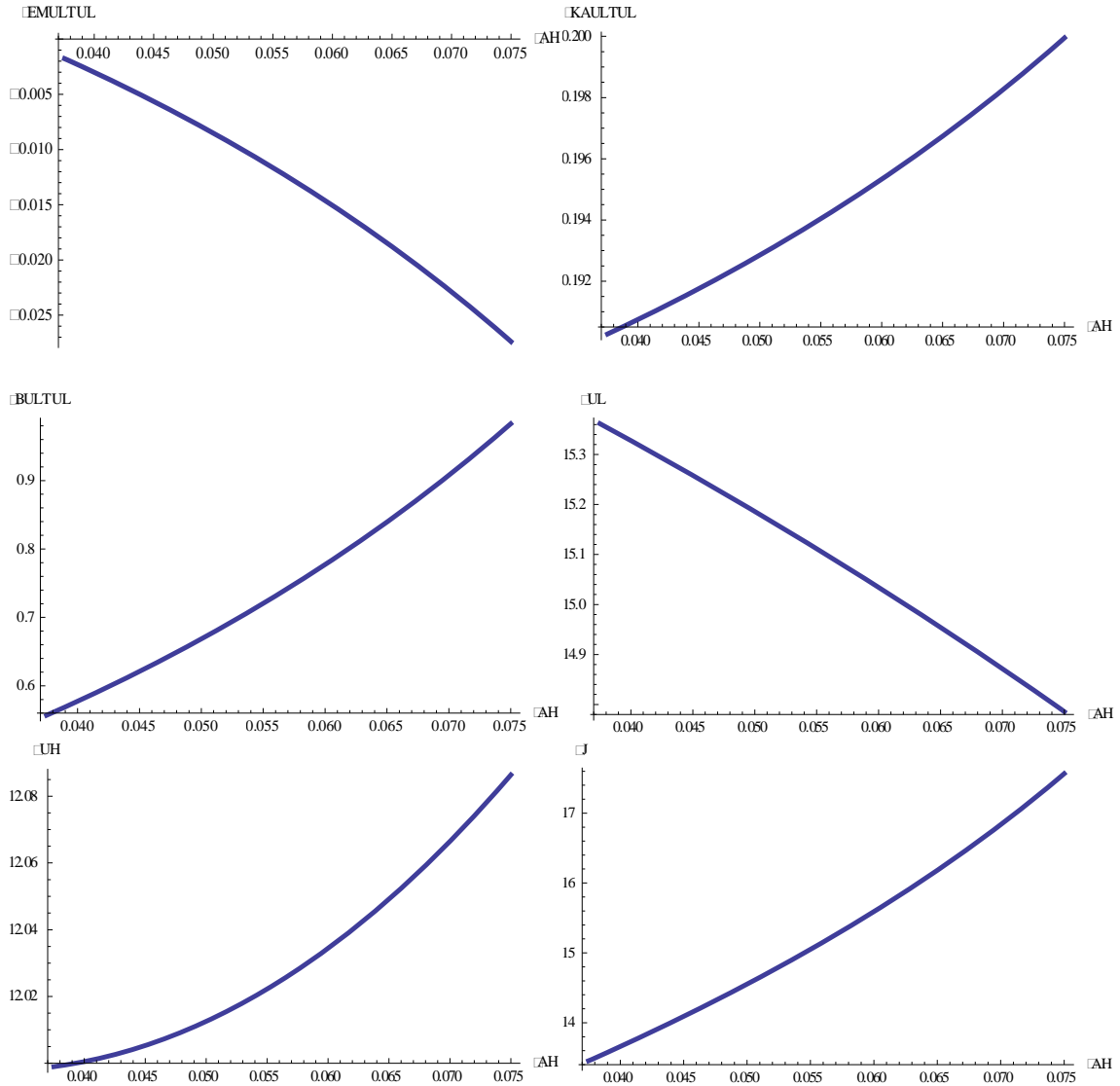

Figure 5.3 Sensitivity results BAM+R&D+UCL

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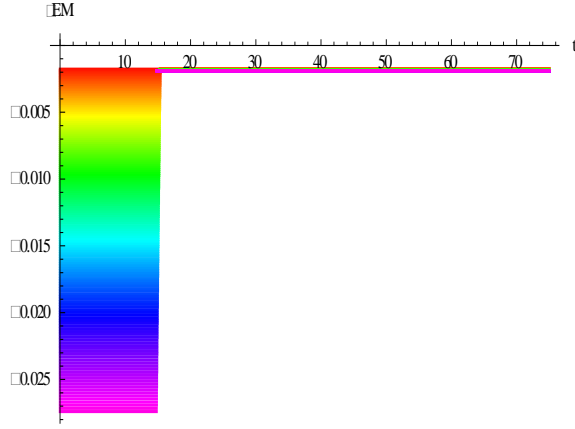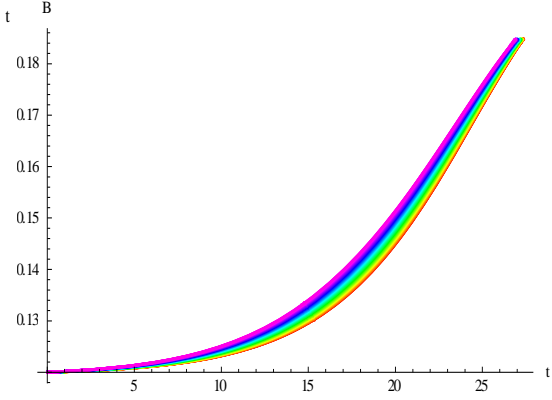

Figure 5.4 Shadow price of CO2 emissions   Figure 5.5 Carbon-free capital productivity, B

Note further, that higher damages also makes it attractive to allocate greater investment resource to R&D activity, whereas, due to the higher expected rates of damages in the future, the effective rate of return on investment in building future carbon-based productive capacity will be reduced, leading to a slower rate of physical carbon-free capital accumulation. The negative effects of this on long-term carbon-free capacity can be mitigated to some extent by making the lower volume of carbon-free capacity more productive – through greater investments in R&D. That is exactly what can be observed from Figure 5.3: one sees that that higher weather related damages tend to rotate the time path of carbon-free capital productivity upwards, while the arrival of the joint production phase is brought forward in time, albeit only slightly.

A higher rate of decay also makes the business-as-usual phase slightly shorter, especially because the low damage sub-phase decreases in length. The latter results from some of the investment in carbon-based capacity during the high damage sub-phase being brought forward in time, as can be seen in figure 5.6.A. In that Figure, we see that the violet time paths are on top of the collection of low damage sub phase time paths, whereas during the high damage sub phase of the BAU phase, they are at the bottom of the collection.[49] A higher rate of carbon-based capital accumulation in the low damage sub-phase ultimately implies a higher volume of the capital stock at the end of the low damage sub-phase. This would lead to a higher volume of capital to be discarded at a later date, unless the accumulation process itself stops earlier than before. Hence, during the joint production phase, carbon-based capital is seen to depreciate considerably faster than in the previous experiments. The forward shift in time of carbon-based net investment shows up as a distinct kink in the high decay parameter time paths. For low decay parameters values, the BAM+R&D+UCL results are very close to the BAM+R&D baseline results that are identical to the reddest time path in figure 5.6.B.

The results for net investment in carbon-free capacity exhibit very different behavior for low and for high damages. When damages are low, the rate of capital accumulation is rising exponentially over time. When carbon-based capital damages are high, the rate of net

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[49] Note that the red time path doesn't show a break at all, because it reflects the baseline of the BAM+R&D model in which there was no difference between the low damage and the high damage sub phase of the BAU phase.

investment in carbon-free capacity is initially higher than in the low damage phase, but at the end of the joint production phase the rate of net investment is even slightly lower than at the beginning. The reason is that due to increased damages of carbon-based capital, the opportunity
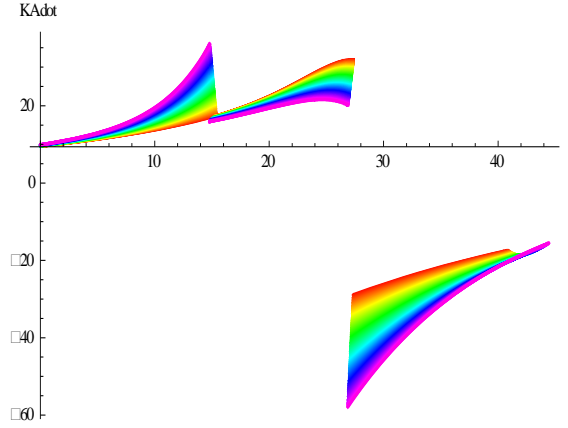

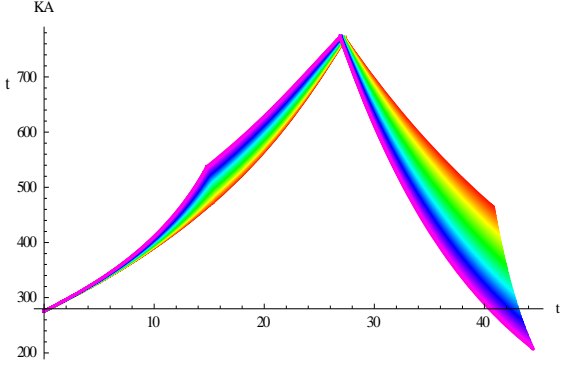Figure 5.6.A $\dot{K}_A : \quad \delta_A^H = 0.0375 - 0.075$

Figure 5.6.B $K_A : \quad \delta_A^H = 0.0375 - 0.075$
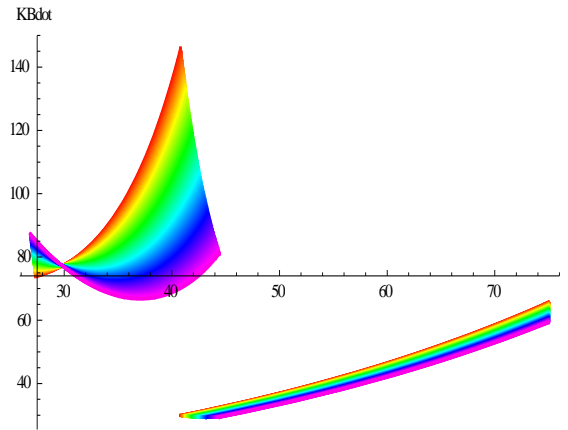

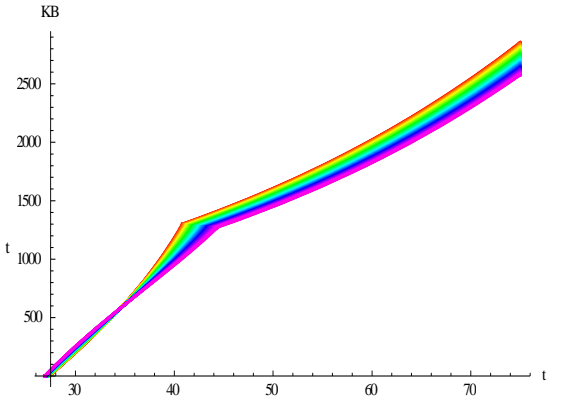Figure 5.6.C $\dot{K}_B : \quad \delta_A^H = 0.0375 - 0.075$

Figure 5.6.D $K_D : \quad \delta_A^H = 0.0375 - 0.075$

The pattern in net investment observed in Figure 5.6.C is reflected in Figure 5.6.D, which shows the slightly higher carbon-free capital stock in the beginning of the joint production phase for the high damage time paths, although after a while those high damage time paths show levels for the carbon-free stock of capital above those for the low damage time paths. Under these high damage conditions, the arrival of the carbon-free phase is postponed somewhat.

From Figure 5.6.E one can see that the extra net investment in the beginning of the business-as-usual phase under high weather-related damage conditions rotates the cumulative emission time path upwards for the largest part of the phases before the carbon-free phase. The rate at which the flow of emissions decreases on the high damage time paths is, however, so large that the upward shift in the beginning of the curve is more than compensated at the end of the curve, leading to an intersection of the high damage cumulative emissions curve and the low damage cumulative emissions curve a few years before the beginning of the carbon-free phase.
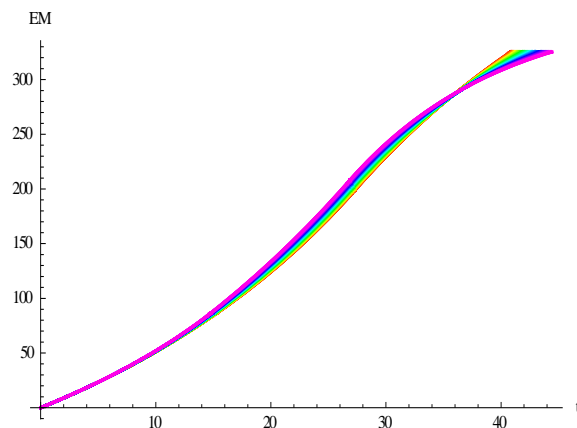
Figure 5.6.E Cumulative emissions E : $\delta_A^H = 0.0375 - 0.075$

The time paths for output are provided in Figure 5.6.F. There is a major disruption when the carbon-free phase begins, but this doesn't really affect the development over time of the corresponding consumption paths, as we have seen before, and now also in Figure 5.6.G. The only slight hiccup in consumption occurs at the end of the high damage sub-phase of the business-as-usual phase.
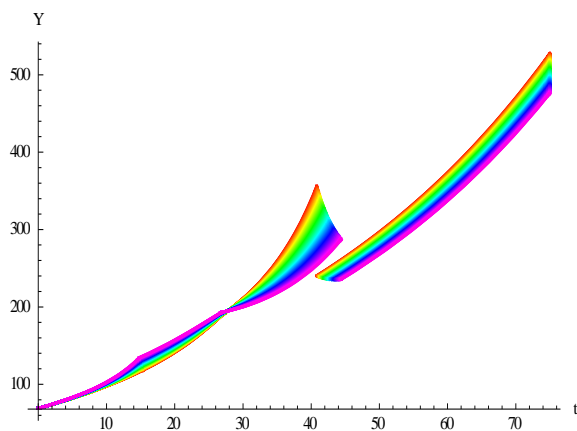
Figure 5.6.F. Output : $\delta_A^H = 0.0375 - 0.075$
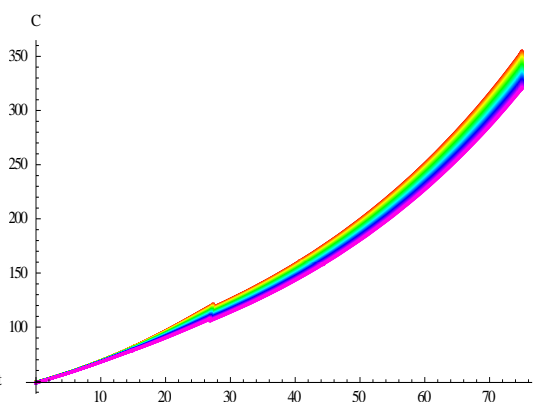
Figure 5.6.G. Consumption : $\delta_A^H = .0375 - .075$
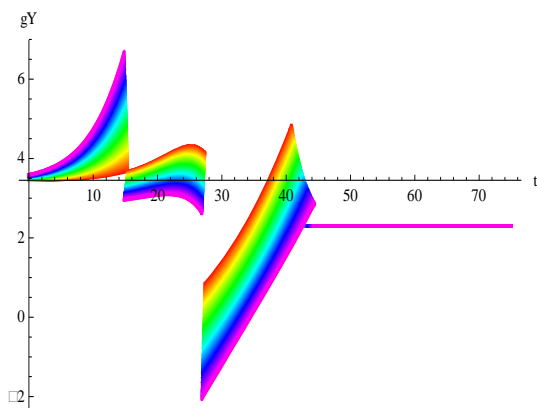
Figure 5.6.H Growth rate $\dot{Y}/Y$

Figure 5.6.I Welfare W: $\delta_A^H = 0.0375 - 0.075$

Figure 5.6.H shows the time paths of the growth rate of output. For the high decay parameter paths the drop in the growth rate of output at the beginning of the joint production phase is much higher than for the low decay parameter time paths. Nevertheless, welfare is hardly affected, as can be seen from Figure 5.6.I.

4 Summary and Some Concluding Comments

This paper has presents a multiphase transition model that can be solved to obtain the optimum timing and magnitudes of tangible and intangible investments that are require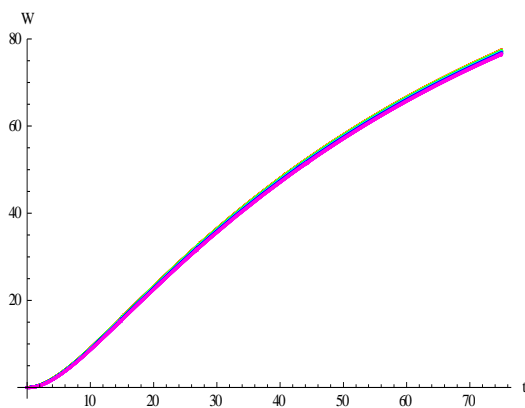d to effect the integrated system's transition from a carbon-based economy to one whose production activities are essentially "carbon-free" -- in the sense of having ceased to generate CO2 emissions exceeding the Earth's natural abatement capacity -- and therefore will have stabilized the global climate system.   That transition is shown to be optimal when it is completed just-in-time, because our modeling approach posits the existence of a catastrophic tipping point in the Earth's climate system. To have failed to stop short of that critical threshold, which is defined for the purpose of our models in terms of an exogenous terminal bound on the atmospheric CO2 concentration level, would have initiated the onset of irreversible runaway global warming featuring abrupt climate changes that would create an environment inhospitable to the survival of civilization, and the greater part of the Earth's human population.

On the latter premise, for which advances in climate science and paleoclimatology now provides a considerable measure of support, asserting the relevance of the precautionary principle for designing an appropriate policy response suffices to recast the economic problem of climate stabilization as that of effecting a welfare optimizing transition from a carbon-based economy to a zero net CO2 emission production regime in time to avert driving the atmospheric concentration to the catastrophe tipping point. A technical virtue of this reformulation is its explicit acknowledgment that the behavior of the coupled economic and geophysical systems beyond the tipping point would be marked by the high frequency recurrence of abrupt discontinuities in temperature and economic and demographic system-dynamics. Proper theoretical modeling would include those non-convexities, and thus vitiate the use of optimal control analysis.

In addition avoiding that particular *cul-de-sac*, this paper has taken an analytical approach that diverges in a second way from the one that has come to characterize research contributions in the field of integrated assessment modeling (IAM) of climate policy. It addresses the problem of finding the sequence of technological options whose development and deployment would be required to stabilize the global climate in the finite but endogenously determined timespan of an optimal transition to an essentially carbon-free regime of production that is compatible with sustainable economic development and continuing economic growth.

A central feature in our approach to that problem is the supposition that the transition towards the desired state of carbon-free production will require a switch in the deployment of alternative capital-embodied technologies. This will entail building carbon-using production facilities that are sufficient to allow the production of enough capital of the kind in which carbon-free technologies are embodied so that the latter can completely displace the troublesome CO2-emitting production facilities. Accordingly, we have opted for a basic growth model setting that features two distinct classes of technology, both of which can produce malleable (homogeneous) output from which either carbon-based or carbon-free capital goods may be formed. Utilizing the former class of production assets, however, results in the release of CO2 emissions in excess of the natural abatement capacity of the Earth's forests and oceans,

thereby contributing to global warming. We assume, not unrealistically, that at the outset of the transition process the "dirty" (carbon-using) mode of production characterized by a higher level of productivity than the "green" alternative: its technologies offer average unit capital costs lower than those for production facilities that embody the available carbon-free technologies.

These two (linear) technology options are introduced into the otherwise simple setting of the AK endogenous growth model by Rebelo (1991), but complicating the latter by the implied requirement that a technical change in the state of the production system requires a specific kind of tangible capital formation to implement the (carbon-free) alternative technology. This is a distinctive feature of the modelling approach pursued here, and rests squarely on the technological premise that *embodiment in physical capital* goods will be necessary to implement switches non-carbon techniques, the increasing relative diffusion of which would tend to lower the global production regime's overall carbon-intensity more rapidly the more closely the average productivity of carbon-free facilities came to approach (and surpass) that of carbon-based capital stock.

The "embodiment" assumption and its implications are especially appropriate in our view, and perhaps more importantly that expressed by engineers with experience-based expertise with innovation processes in energy supply systems. Yet, our explicit recognition of this constitutes an important departure of the present analysis (and its account of the dynamics of the optimal climate-stabilizing transition path) from the way in which the effects of endogenous technological change have been treated in previous economic contributions to integrated modelling of climate policy measures – where assuming that technical change is disembodied has long remained.

Focusing on sequencing the deployment of different categories of technology policy, our imposition of minimally restrictive efficiency conditions leads to a multi-phase optimization framework in which the Basic Model of technology switching recognizes three separate phases of production, each of them characterized by different technologies or combinations of technologies being active. It is shown that due to the linear production technologies employed there will always be an active tangible capital formation process to implement one or the other of these technologies -- even though output can be produced using both technologies concurrently as long as there is still room to emit CO2 from the facilities that use energy obtained by burning fossil fuels. The first of the three phases is called the 'Business-as-usual' or BAU phase, in which carbon-based capacity is still being built up, and the output produced is completely carbon-based. At some point in time the next phase arrives, in which investment in carbon-based capital ceases and the buildup and concurrent use of carbon-free capacity begins. During that phase, production using carbon-based capital continues but the production level falls over time as the capital stock is worn down due to technical decay. The second phase is called the joint production phase (JPR phase for short), as both technologies are used to produce output. At the beginning of the third phase, called the 'carbon-free' phase (CFR phase for short) the remaining carbon-based capacity is scrapped and production is from then on completely carbon-free: the green future has arrived (just-in-time).

We extend the basic three-phase BAM setting in two ways. First we introduce endogenous R&D driven technical change that improves the productivity of the carbon-free technology up to the moment of the installation of the first unit that embodies the new technology. The reason to do this is that the embodiment of technology in fixed capital implies that the buildup of significant stocks of carbon-free capital will take time which may turn out to be in short supply on the one hand, while on the other hand it will also draw upon the existing carbon-based capacity at the expense of consumption, which is the sole source of welfare in the

standard AK-model and in our model too. To keep matters simple, we make the assumption that R&D driven technical change stops the moment the new technology starts to be implemented.

The build-up of carbon-free capacity to a level that is sufficient to mitigate the aggregate productivity drop associated with the scrapping of remaining carbon-based capacity at the beginning of the carbon-free phase will take time. That fact is one of the main reasons for focusing attention on the optimum timing of the changes between the different phases of the model. The specification of the way R&D alters the productivity of capital embodying carbon-free technologies, when incorporated in an AK-setting with a limiting constant marginal product of capital, generates productivity levels for the class of "alternative" technologies that are bounded from above (asymptotic technical change) -- reflecting the 'fishing-out' effect recognized in much of the endogenous growth literature since Jones (1995). The latter effect seems especially relevant in regard to the further development of a specific class of technologies, as opposed to innovation and realized technical changes occurring at the macro-level as consequence of   expansion of technical constraints affecting a broad and diverse array of goods and processes.

The second extension of the BAM model pertains to the introduction of a weather related damage threshold. For this purpose we proceed simply, by making the assumption that high rates of loss of productive capacity due to expected, but unscheduled rates of loss in productive capacity will occur after the system reaches an "extreme weather tipping point" (defined in terms of mean global temperature, or, equivalently, a CO2 concentration level). The latter is set below the climate catastrophe tipping point, and triggers an upward jump in the expected proportional rate of technical decay of the capital stock. The effect is tantamount to raising the annual rate of physical depreciation of the capital stocks.

The timing of the phase changes present in the various model versions are governed by transversality conditions following from the optimality condition that the values of the Hamiltonians must be identical when evaluated at the moment of the phase change under the conditions pertaining to the phases just before and just after the passage between phases. In the Basic Model (without R&D and extra weather related damages) the condition defining the optimum length of the BAU phase turns out to be the requirement that the shadow prices of carbon-based and carbon-free capacity should be the same at the moment investment in the latter technology takes over from investment in the former. This makes perfect economic sense, as the opportunity cost of a unit of investment in terms of consumption foregone is the same in both cases. For the timing of the change from the JPR phase to the CFR phase, the optimality condition results in a transversality condition that states that the moment that the CFR phase should begin is defined by the requirement that the benefits of continuing to use carbon-based capacity (the utility value of its output) is outweighed by the cost of doing so (the (negative) utility value of the corresponding emissions), which closely resembles the 'negative quasi-rent' scrapping condition known for a long time now from the vintage literature with fixed factor proportions ex post (i.e. putty- clay and clay-clay models).[50] For the other versions of the model, more complicated transversality conditions arise out of the same general optimality conditions pertaining to the equality of Hamiltonians at the moment of a phase change.

BAM contains two further transversality conditions, i.e. the standard one associated with the pure AK-setting of the CFR phase[51] and the one implied by the fact that from the start of the

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[50] Cf. Johansen (1959) and Solow et al. (1966).

[51] In the endogenous R&D version of the model, still another transversality condition is that as time goes to infinity, the present value of the welfare value of carbon-free capital productivity should approach zero.

CFR phase the shadow price of carbon-based capital should be zero, as carbon-based capacity is worthless from then on. Using the requirement of the continuity of state and co-state variables at the moment of a phase change, the transversality conditions in combination with the given initial and terminal values for the state variables in the model allow us to use a (steepest descent) search method that, for *a priori* "guesses" of the still missing initial values of a subset of co-states and of the phase lengths of the BAU and the JPR phases, solves the systems of differential equations resulting from the first order conditions of the Hamiltonian problems for each of the individual phases. Given this solution that is contingent on the *a priori* initial values, we can evaluate to the extent to which the different transversality conditions and the boundary condition for cumulative emissions are met. If one or more of these transversality conditions or the boundary condition are not met, the initial value(s) need to be adjusted; if all the conditions are met then the optimum path has been found.

We have calibrated the parameters and initial values of the BAM model, based to a large extent on Nordhaus' 2010 (DICE and) RICE model dataset and added some a priori assumptions regarding the capital productivity parameters and the parameters of the R&D function. Using this setup, we have run a number of simulations to investigate the sensitivity of the optimal solutions of the various model versions to changes in the structural parameters featuring in the utility function and in the production functions, which all showed the expected outcomes known from the standard endogenous growth models.

We then turned to a sensitivity analysis that involved the location (on the scale of atmospheric CO2 concentration levels) of the tipping point for irreversible runaway climate change, and the lower tipping point for the onset of greater expected rates of extreme-weather damages to productive capacity. In the simulation where we systematically reduced the climate change threshold in order to see what a more stringent application of the precautionary principle would mean, we find that in BAM it will be optimal to accumulate carbon-based capital at a faster pace than with a less tight cumulative emissions threshold, so that the arrival of the CFR phase is speeded up. To facilitate the latter, a quick buildup of the carbon-based capital stock is required to be able to switch relatively early to investment in carbon-free capacity and to enable considerable rates of both investment and consumption once investment in carbon-based production and later on production using carbon-based capacity has ceased. Tightening the cumulative emissions constraint raises (in absolute terms) the shadow price of CO2 emissions, but also that of carbon-free capital. But somewhat unexpectedly perhaps, it also raises the shadow price of carbon-based capital, simply because *the value attributed to the carbon-based capital stock is for an important part derived from the value of the carbon-free stock that it is able to produce.*

When we allow for endogenous R&D reactions in this setting, we observe that the length of the BAU phase increases relative to the BAM case. The latter allows R&D activity to be more evenly spread over time, which, due to the concavity of carbon-free capital productivity in R&D efforts raises the overall effectiveness of a given R&D budget. There is also more time to accumulate carbon-based capital, so that at the end of the BAU phase there can be a considerable terminal value of carbon-based capital: the extension of the BAU phase allows the economy to eat its carbon-based cake and still have a considerable amount of it left at the beginning of the JPR phase in the form of carbon-based capital. In contrast to the BAM case, the JPR phase is now shortened as the cumulative emissions threshold becomes tighter. The overall effect is that the CFR phase comes slightly earlier than in BAM, while the dispersion in the welfare effects is less than under BAM: *the possibility to change productivity through R&D helps to fight the negative welfare effects of tightening emission constraints.*

Two quantitative aspects of the optimal transition paths found for these three models should not pass without notice here: both aspects are exhibited in the following comparative compilation of the durations found for the phases of the climate stabilizing transition. The first and perhaps the most striking feature of these results is the close consilience of the length of the completed transitions that emerge from the welfare-optimizing solutions of the different models. As seen from the bottom-most line of the tableau, the durations of the optimized transition remain tightly clustered around the 40-year mark, the underlying shifts in BAU and JPR phases' absolute and relative lengths notwithstanding

Comparison of Alternative Models' Optimal Phase Durations (years)				
Models: → *Phases:*	BAM+R&D+UCL	BAM+R&D	BAM	{BAM+R&D} – BAM *Effect of R&D:*
Business-as-Usual: *BAU*	27.36	27.36	23.80	+ 3.56
Joint Production: *JPR*	12.38	13.45	16.39	- 2.94
Completed Transition: *BAU+ JC*	39.74	40.81	40.17	+0.64

Source: See text, above: sub-sects. 4.3 para.1 and 4.4 para. 2.

These numbers give a useful degree of concrete-ness to the point made qualitatively in Section 4's discussion of the effect that introducing the R&D option has in extending the business-as-usual phase, while compressing the time taken (during the joint production phase) to implement of the actual switch from carbon-using to carbon free production capacity. Comparing the results for BAM with those for BAM+R&D in order to gauge the effect of exercising the R&D investment option, the right-most column shows the largely offsetting movements of the two component phases. Introducing R&D investments stretches out the optimal duration of the BAU phase, vis-à-vis that found with the Basic Model. But, their effect in raising the average productivity (and, by that token, lowering the unit capital costs) of the subsequently installed carbon-free capital stock, in combination with the somewhat larger carbon-using capital stock that can be built up during the extra 4.4 year, permits a faster switch-over during the JPR phase. R&D thereby lessening the transition' toll on consumption and the level of social welfare.[52]

It is important to emphasize the general insight that the welfare-enhancing effect of introducing R&D activities in this planning analysis framework, and, more generally of optimally deploying still other options from the larger and diverse portfolio of available techniques that will mitigate CO_2 emissions, are property that is the dual of the essential constancy that is seen in the endogenously determined transition durations. This is not to say that either the specifications of the several models or their calibrations were selected with a view to yielding the common properties of their optimal solutions that have been highlighted here.

The clustering of the lengths of the transitions in the neighborhood of the 40 year mark, for this group of models has a cause that may be grasped intuitively. Each model starts from the same implicit level of atmospheric CO_2 concentration and has the identical "budget" for

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[52] The identical durations of the BAU phase in the models with R&D added to the BAM model, whether or not allowance is made of weather-related "unscheduled capacity loses" (UCL) reflects the assumption that R&D activities do not entail CO_2 emissions, and thus do not directly contribute to exhausting the budget of allowable emissions.. Further, while Anticipated capacity losses from in the "high damage" sub-phase of BAU actually will lower the the annual flows of output and emissions, additional carbon-based capital formation over BAU phase compensates for that. To the extent that R&D does use fossil fuel-based high temperature ovens and electricity supplies, the rising trend of final goods and services available for consumption is overstated, especially in the latter years of BAU.

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permissible further cumulative emissions –because the social planner must in every case complete the transition before reaching the same "tipping point" (at c. 447 ppmv), in order to avoid exceeding a global mean temperature gain of 2° Kelvin above pre-industrial levels. The uniformity of the "hard budget constraint" in regard to net emissions generated during the transition is a property of the common climate system that one must posit in order to render the optimization results comparable. Obviously, this condition holds for any [start, stop] pair of CO_2 concentration levels. Therefore, the way to see the essentials of what is going on in these DIRAC transition path-optimization problems is this. The social planner will maximize the global present value of economic welfare by taking whatever actions are required to spend that budget efficiently in exploiting whatever technological means already exist or can be made available. But she always will contrive to have completely exhausted the fixed emissions budget just before the economy enters the phase of zero (net) emissions. Given the anticipated constraints imposed upon optimization by the dynamics of the climate system, better technological tools therefore will be translated into greater real income and increased social welfare. Of course, if the emissions budget already had been squandered before the planner arrived on the scene, the game might well be over before any meliorative actions could be taken.

The main purpose in building and analyzing models such as those presented here is to expose and illuminate the dynamic interrelationships entailed in those actions, rather than to predict the required duration of the completed transition, or of its constituent sub-phases. But, coming now to the second striking aspect of the models' outcomes in that regard, one cannot help but be surprised (and not a 'tickled') to find that the "required" transition period on which they concur lasts four decades from the t=0 point, which our calibrations set to correspond with 2010 (see, above, footnote 41). Thus, our models turn out to be aligned in that respect with the conclusions of recent comprehensive studies of the feasibility and extent of the technological and physical transformation in modern energy systems, which have focused 2010-2050 as the period during which a concerted technological effort could manage to end the contribution of anthropogenic GHG emissions to global warming.[53]

When we take account within this framework of the greater expected frequency of warming-driven "extreme weather" damages to the flow of real output from vulnerable productive capacity, such that the rate of unscheduled losses of K_A is irreversibly shifted upwards at some point during the BAU phase, an extra phase is added to the Basic Model's three phases. The extended version of the Basic model then has a low damage BAU phase, a high damage BAU phase, and (high damage) JPR and CFR phases.[54] The weather related damages

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[53] For example, the National Research Council (2010) report of its Committee on America's Energy Future made detailed technical assessments of the energy-supply and end-use technologies that were judged most likely to have meaningful impacts on the U.S. energy system during the three time intervals: 2009–2020 for(deployment of existing technologies, 2020–2035 for development of new technologies for commercial deployment), 2035–2050 for further development and widespread deployment of advanced technologies (cf. Full Report, pp.134-135. A member of the AEF Committee, Robert W. Fri (2013:p.6), remarking more recently on the scale and complexity of the U.S. energy system, has stressed the point that "a few decades is not a very long time to overhaul it to the point where it emits essentially no greenhouse gases"....But it is difficult to come up with a high-probability scenario that does not exhaust the emissions budget by 2050." We view these coinciding references to 2050 as the transition's endpoint to be little more than "coincidental."

[54] This admittedly simplistic specification – which avoids creating still more phases (needed to accommodate a number of discrete upward steps in the capital loss rate) may be rationalized along the following lines. The location of fixed infrastructure facilities in the BAU will be legacies of a previous era in which sea levels and moisture levels in the atmosphere were lower, and existing K_A will not have been designed to reduce their vulnerability to the distribution of extreme weather events that will ensue from the warming that already has occurred. Moreover, further additions to

introduce an additional incentive to engage in R&D. The reason is that the weather-related damages pertain to physical capital, while the productivity of unaffected units of capacity (and more importantly that of the carbon-free capital stock to be built in the future)will remain untouched. Consequently, in these anticipated circumstances, investments directed to raise the productivity of carbon-free capital is a natural means of maintaining a reasonable net capital formation. In addition to the extra R&D activity, we also observe extra investment in carbon-based capacity during the low damage BAU phase. The reason is that the expectation of extra weather-related damages introduces a wedge in the rate of return on carbon-based investment during the low and high damage sub phases of the BAU phase. An increase in the extra damages rate also leads to a more even distribution of carbon-free investment over time, in order to mitigate the negative effects on consumption of increased weather related damages. With low damage rates, investment in carbon-free capacity during the JPR phase is strongly increasing in anticipation of having to cushion the drop in output associated with the scrapping of the remaining carbon-based capital stock at the beginning of the CFR phase. With higher damages, the drop will necessarily be less, *ceteris paribus*.

Concluding remarks:

So, what are the main lessons to be drawn from the foregoing welter of specific details about the optimal transition dynamics displayed by these very simple models of a global eco-climate system? First, one must stress the general point that the embodiment of technical change in physical units of capital underlines the practical importance of the notion of capital as a produced means of production. Recognition of this points to the need to build or maintain a carbon-based capital production system that is adequate to produce the right amount and quality of the carbon-free production units on which future welfare will exclusively come to depend; and to do so within the remaining time-span allowed by the accumulating stock of atmospheric CO2. The results also draw attention to the implication that taking into account embodied technical change in the design of a "tech fix" climate policy a worryingly short duration for the BAU phase.

Secondly, R&D emerges from this analysis as an important means of cushioning the negative welfare effects of tighter emission thresholds and increasing weather related damages. The reason for this is that physical capital investment in carbon-based capacity is at the same time both a substitute for and a complement of R&D investment. It is a substitute from a pure production point of view, while it is a complement because of the embodied nature of technical change that needs physical investment to turn potential productivity improvements into real ones. Because the return to R&D and physical investment in carbon-free capacity depend positively on each other, output itself, and therefore consumption and investment possibilities are positively affected by having the possibility of engaging in R&D. Hence, R&D efforts provide an additional means to both reduce the dispersion in welfare outcomes and to maintain or even increase future carbon-free output levels and growth rates in the face of increasingly volatile weather events and corresponding damages and a rising probability of runaway global warming.

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the stock of carbon-based infrastructure are likely to continue to be sited essentially at the same types of locations, to exploit existing engineering designs lower costs of servicing existing directly productive capacity. While K^B , being newly formed, could be sited so as to be less vulnerable to the weather variations in the altered climate conditions, uncertainties combined with greater costs could combine with increasing severity of storms and flooding (at still further elevated mean global temperature) not only throughout the JPR and CFR phases. A later specification, however, will do away with the notion of one (or more) temperature trigger points for increased expected capital losses, by making the latter a continuous function of the endogenous level of mean global temperature.

Thirdly, the idealized "optimal planning" framework for endogenous macroeconomic growth seen to be well suited to incorporating representations of the technical aspects of the array of existing and potential technological options and their respective resource requirements, as well as those required to operationalizing a welfare-optimal transition path. Application of multi-phase optimal control analysis provides DIRAM solutions that describe the optimal flows of tangible and intangible capital formation, along with the production flows using carbon-based or alternative technologies in each of the phases, as well as the sequencing and respective durations of the latter which completely describe the completed transition path.

This very concrete way of setting out what has to be accomplished technologically in each of the phases, in our view, can provide a useful starting point for thinking about how to design and coordinate the multiplicity of diverse tasks that will need to be undertaken in order to adequately respond to the daunting existential challenges posed by global warming and climate instability. More intricate and computationally demanding modeling and analyses of temporally extended multi-phase transition paths surely will be necessary to shed greater light on the complexities entailed in working out the proper dynamic sequencing for the integrated development and exploitation of the variety of complementary and competing technology policy options -- beyond the very few that have figured in the preceding pages. Among the myriad "of other options" that remain to be considered, the following handful deserve priority positions on the agenda for continued future DIRAM research – by virtue of their varied functional attributes:

(i) harvesting "low-hanging fruit" -- by investments that implement core engineering know-how to incrementally retrofit existing carbon-burning infrastructure and direct productive capital goods, allowing these to switch to the "less dirty" fossil fuels as energy sources (e.g., natural gas, in place of oil for heating and (diesel) internal combustion engines), and/or upgrade the energy efficiency of existing buildings and production equipment, and the same might well be said for curtailing releases of methane and other potent GHGs;

(ii) R&D expenditures on risky exploratory research programs that have longer-time horizons than the norm for applied projects (such as those considered here) because they seek low-frequency "break-through" discoveries and inventions that would drastically lower the unit capital costs of reliable production facilities that use alternative, non-carbon energy sources;

(iii) Long-term resource support for experimental geo-engineering projects that work in parallel on the development and field-testing of safely scalable "back-stop" technologies for atmospheric carbon capture and sequestration (ACCS), and locally deployable solar radiation management (SRM) techniques.[55]

(iv) Capital formation for reforestation with fast-growing leafy trees in order to efficiently raise the natural capacity CO_2 abatement capacity of the Earth's present forest cover as far as possible. Although it is highly unlikely that this measure could compensate for the degraded abatement capacity of the oceans that would result from continuing warming,

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[55] Although under the suppostions of our deterministic optimal control models of the climate-stablizing transition path those techniques never would need to deployed in a "back-stop" role, even were to some among them to be economically as well as technically feasible and environmentally safe to implement. But in the latter circumstances the investment in finding those method could nevertheless have a substantial social pay-off, especially when the transition to a low carbon global production regime had to be completed in reasonably short order. In that case it is most likely that a considerable stock of operational carbon-based capital would have to de-activated before the sustainable economic growth phase began. But, by deploying ACS techniques that were (*ex hypothesis*) technically effective and economically practical, the concentration level of CO_2 could be gradually lowered far enough to allowing the "moth-balled" carbon-based production facilities to be brought back into production, thereby yielding a finite flow of social quasi-rents. A fast transition also would imply that the latter's present value would not have been so severely reduced by discounting. Of course, in a stochastic control setting, "back-stop" geo-engineering investment would have positive insurance value even were the research results to turn out never to be needed, or deemed too risky in their potential environmental side-effects to be deployed only to capture the private quasi-rents..

pursuing this option should be seen not as "a fix" but as a "buy time" strategy (similar to item (i) above) in its effects on the net volume of CO2 that is added to the atmosphere by burning carbon. Policy measures aimed at slowing or actually halting the clear-cutting of forests would work in the same direction, and might also involve compensatory capital formation to raise agricultural yields and the livestock carrying capacity of already cleared lands, thereby tending to reduce the economic pressures that are driving deforestation due to human agency.

(v) "Defensive" expenditures for engineering design research and capital formation to implement physical reinforcements and additions to existing infrastructure that would reduce the latter's own vulnerability to damage caused by extreme weather events and extensive flooding. This would serve to mitigate direct and indirect losses of productive capacity as well as providing a measure of protection from loss of lives and livelihoods in some forms of natural disasters that are likely to become more destructive due to regional climate changes.

It will be no mean task for future research to develop informative heuristic representations of the foregoing dynamic processes, and explore the way(s) to sequence the exercise of the enlarged set of policy options in an integrated multiphase model that would extend the partial "BAM+(applied)R&D+UCL" structure that has been analyzed (in section 4.4).[56] Moreover, it is a task that will be rendered more feasible by tackling it within the simplifying framework of a "social planning model" that greatly reduces the number of assumptions that need to be made (and empirically justified) -- about the market behaviors of private economic agents, and the ability of political authorities to coordinate and implement coherent regulatory measures and institutional enforcement mechanisms on a global scale.

Fourthly, having appreciated the difficulties of carrying through such an undertaking, it is all the more important to appreciate that its completion would stop well short of the further steps that would lead towards articulating the available practical policy instruments, and the possible ways they might be used to reach successive milestones on the DIRAM multi-phase path that an omniscient and omnipotent social planner would indicate as "required" to achieve the optimal transition. That poses a research problem far more daunting than anything that we have contemplated tackling; indeed, one that may well resist plausible formulations for solution using optimal control methods (whether of the deterministic or stochastic variety). To the already complex macroeconomic and geophysical system constraints considered in a DIRAM framework, it is now necessary to add further constraints imposed by public sector political and administrative processes, and, of course, the behavioral patterns of private economic agents in response to market signals and regulatory inducements and restraints.

Moreover, it is to be expected that the design of the optimal inter-temporal policy implementation program – were one to emerge from the analysis, would depart substantially in many respects from the requirements indicated by the social planning optimum. Because it is dealing with a process to be played out in major part within a decentralized market setting, the implementation program would involve recourse to the use of policy instruments in addition to (inevitably interactive with) public-private joint ventures and government led "technology push" measures. The plural here is inescapable in the relevant practical context of a global

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[56] A start was made on this research agenda by David and van Zon(2012:section 3), in working out the phase structures of the transition path(s) for a model that integrated "BAM + R&D" with a "buy time" option that involved incremental "retrofitting investments" (for applied engineering and capital equipment) to upgrade the environmental performance of carbon-using capital. The specific functional purpose of the latter class of options is to lower CO_2 emissions per unit of output in the selected facilitiesy, even though the upgrade had raised their unit capital costs. Preliminary computational results for the resulting (uncalibrated) optimal sequencing were presented for that option. Furthermore, that paper sets out a list (in section 4) of desireable modifications in the modeling of the climate system, of the production system, and other specification changes. The preceding text has offered a shorter and less detailed sketches in the same spirit.

system with multiple sovereign political authorities--however far one can hope to go by abstracting from the messiness of the processes by which national policies emerge from often chaotic domestic conflicts among conflicting politically mediated interests.

Fifthly, with this situation in mind, it might be a useful way forward for practical policy design research to focus on illuminating how and to what extents the introduction of ostensibly "practical" means to implement the indicated allocation of resources (within each of the successive phases of the optimal transition path found by the DIRAM planner) would distort the allocation that materialized in the first phase, making it necessary to alter the indicated requirements for the phase that would follow, and so on. In principle, at least, the distortions and the welfare losses entailed by a well-specified policy implementation program could be calculable. This step on the path toward stochastic control modeling that allows mid-course corrections when expectations are not realized, however, generally would require taking an intervening and non-trivial step: fully specifying a model of the behavior of private sector actors in response to the array of relative price and cost changes, and subsidies, taxes and penalties introduced by fiscal and regulatory measures designed by a public actor in order to implement the policy in question. Further allowance would need to be made for adjustment costs of alterations in pre-existing tax, subsidy and regulatory structures, including such new institutional developments that would be necessary.

Consequently, although it is interesting and informative to trace the complex inter-temporal adjustments that are set in motion by adding to the variety of technological options and corresponding control variables in the heuristic DIRAM framework presented in these pages, the policy relevance of showing how an enlightened social welfare-optimizing planner might run a "tech fix" program to stabilize the global climate system under various technological and climatic system constraints is bounded by the relevance of the circumstances envisaged by those models. The extent to which the planner's optimal program offers practical guidance for the design of climate policies that is of use to policy–making by individual actors and agencies in the world as it is, remains an unresolved question. Yet, such guidance will be needed, even if it is not presently wanted by decision-makers and public actors and agencies that lack anything approximating the mythical social planner's powers to shape the allocation of resources; they will have to cope as best they can by applying indirect and comparatively weak and uncertain instruments in highly decentralized political and economic settings.

We close these comments by remarking on a particular aspect of the DIRAM transition results presented in section 4 that is common to the solutions found for each of the models, and which may serve to underscore the need for interpretive caution and restraint from overly casual drawing concrete "policy insights" from heuristic exercises such as those conducted in this paper. In every case, as was noticed, the optimal time-path found for the co-state of E(t) -- which is to say the (negative) marginal values of current period flow of emissions that will add to the atmospheric concentration of CO2 – appears as a horizontal line throughout the transition, up until its completion and the beginning of the carbon-free phase of economic growth. Applying the label "social cost of carbon" is an entirely valid way to interpret the economic welfare significance of that "shadow price" (marginal valuation) of the difference between the currently stock of atmospheric CO2 and its pre-industrial level --that being the way the state variable $E(t)$ is defined. But it would be entirely inappropriate to casually take the further interpretive step and think of the social cost of carbon found from the solutions to the stacked Hamiltonians in the present analysis as indicative of the size of "an optimal carbon tax."

That interpretation is a perfectly valid step and one that is taken as a matter of course in (IAM) climate policy assessment exercises of the kind that have been made familiar by the DICE and RICE models applied developed by Nordhaus (e.g., 1994, 2000, 2007, 2010), and other integrated assessment studies that have drawn directly and indirectly on his pioneering contributions to that approach. But it doesn't follow in the present context for a simple reason: the DIRAM structure developed here differs radically from that class of IAM structures, most obviously in not having considered the optimal design of climate policy instruments such as scheduled taxes on fossil fuel materials, or market-determined prices for tradable licenses to release volumes of CO_2 generated by burning carbon. The shadow prices of the stock of atmospheric CO_2 on the optimal "tech fix" transition path do not describe the time-path of a control variable in the form of a tax, and so their behavior is without any *intrinsic,* independent significance for the design of such policy instruments.

It should be understood, instead, that the constancy of the marginal social cost of carbon emissions along the transition paths leading the phase of carbon-free production in our models is merely the formal reflection of the fact that throughout that span to time the cumulative stock of CO_2 (by construction) has been kept from reaching the (pre-specified) "tipping point" into "climate-system catastrophe," The marginal social costs that are entailed at each moment until the economy is safely in a viable carbon-free state are those measured in terms of the unchanging value of the "felicity" foregone by having to divert enough output from consumption to other (investment) activities that will avert adding the marginal unit of CO_2 the existing atmospheric concentration of greenhouse gases.

Future analysis of the problem of policy design with portfolios of distinct technological options should certainly move towards building models in which carbon-pricing mechanisms are available for use as policy instruments, and apply such modes to the analysis of "mixed control systems" that use those tools, along with a variety of other means by public policy and private non-profit initiatives that would contribute to speeding costly transformations of the global regime of production in the world's "mixed economies." That task would seem to open a temptingly large and policy relevant field for systematic explorations using multiphase optimal control techniques. Yet, at present it appear to be one that may be tackled more productively once the frontier of DIRAM research has advanced far enough to permit full characterizations of the optimal transition paths for technological system-settings that afford a richer, and considerably more complicated portfolio of options than those considered in this preliminary exploration.

Appendix A. The Timing of R&D

Consider the availability of $B_0 > 0$ at t=TU. Consider also the possibility of waiting a while, say till T*, before starting R&D activities while still relying only on A-type production facilities. . Then at T* the BAU phase would be effectively split into two sub-phases, first U0 followed by U1. Assume, moreover, that the split at T* is optimal. In that case, the only difference between these sub-phases involves the existence in U1 of an R&D process that needs resourcing from some portion of current production to raise the average (and marginal) productivity of carbon-free capital, B, above level (B_0) available with technology of that kind that was already known at the start of the BAU phase. So, the Hamiltonian at T* for sub-phase U0 would be given by:

$$H_{T*}^{U0} = e^{-\rho \cdot t} \cdot C_{T*}^{1-\theta} / (1-\theta) + \lambda_{T*}^A \cdot \{(A - \delta^A) \cdot K_{T*}^A - C_{T*}\}$$ (A.1)

Since the first order condition for consumption can be solved for C, giving rise to $C_{T*} = f(\lambda_{T*}^A, T*)$, the Hamiltonian can be rewritten as:

$$H_{T*}^{U0} = g(f(\lambda_{T*}^A, T*)) + \lambda_{T*}^A \cdot \{(A - \delta^A) \cdot K_{T*}^A - f(\lambda_{T*}^A, T*)\}$$ (A.2)

For U1, we therefore have the Hamiltonian:

$$H_{T*}^{U1} = g(f(\lambda_{T*}^A, T*)) + \lambda_{T*}^A \cdot \{(A - \delta^A) \cdot K_{T*}^A - f(\lambda_{T*}^A, T*) - R\} + \lambda_{T*}^R \cdot \varsigma \cdot R_{T*}^\beta \cdot (\bar{B} - B),$$ (A.3)

and find that the first-order condition for an optimal allocation of R&D resources implies that

$$\lambda_{T*}^A R_{T*}^\beta = \beta \cdot \lambda_{T*}^R \cdot \varsigma \cdot R_{T*}^\beta \cdot (\bar{B} - B).$$ (A.4)

Hence, for there to be a positive level of R&D activity taking place from T* on, it must be the case that:

$$H_{T*}^{U1} = H_{T*}^{U0} + (1-\beta) \cdot \lambda_{T*}^R \cdot \varsigma \cdot R_{T*}^\beta \cdot (\bar{B} - B) > H_{T*}^{U0}$$ (A.5)

It follows that $H_{T*}^{U0} - H_{T*}^{U1} < 0$ for $R_{T*} > 0$, which implies that T* was set too late in the BAU phase and should be moved to an earlier date. [57]. In this case, because that difference between the two Hamiltonians would not vanish, that adjustment process would have to push the arbitrary start point of the U1 sub-phase all way back to T*= TU, i.e., to the beginning of sub-phase U0. That implies that the R&D process optimally would start at the moment that the basic idea(s) became available to how to conduct research and development on a practical carbon-free alternative to the A-type technology, i.e., one that could be embodied in capital goods with positive productivity. Note that we have assumed that $B_0 > 0$ at t=TU=0.

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[57] By reducing T* by 1 unit of time, i.e. let the U1 phase begin one unit of time earlier, we lose the Hamiltonian at the end of U0 and gain the Hamiltonian at the beginning of U1, which in this case would lead to a net gain.

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