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Social interactions and complex networks

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Daniel C. Opolot†

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Abstract

This paper studies the impact of interaction topologies on individual and aggregate behavior in environments with social interactions. We study social interaction games of an infinitely large population with local and global externalities. Local externalities are limited within agents' ego-networks while the global externality is derived from aggregate distribution in a feedback manner. We consider two forms of heterogeneity, that due to individual intrinsic tastes and that due to ego-networks. The agents know the potential number of other agents they will interact with but do not posses complete information about their neighbors' types and strategies so they base their decisions on expectations and beliefs. We characterize the existence, uniqueness and multiplicity of equilibrium distribution of strategies. By considering arbitrary interaction topologies, we show that the interaction structure greatly determines the uniqueness and multiplicity of equilibrium outcomes, as well as the equilibrium aggregate distribution of strategies as measured by the mean strategy.

Keywords: Complex networks, Partial information, Local externality, Global externality, Adoption.

JEL codes: C72, D82, D84, 033

1 Introduction

Social influence plays a great role in a wide range of socio-economic environments. Its effect on individual behavior was recognized as early as 1900's by sociologists such as Georg Simmel in his seminal work on social types; for example Simmel [1904], where he points out the role of social influence in fashion and fads. Social influence, or commonly referred to as social interactions, is also seen to take part in trade and political alliances, public good provision, crime and technology adoption, evolution of scientific theories, labor participation and employment among others. Empirical work has been done to identify and quantify the role of social interactions in crime

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[Glaeser et al., 1995], labor participation [Mulligan, 1998], out-of-wedlock pregnancy [Akerlof et al., 1996], unemployment [Topa, 2001] and substance use [Jones, 1994]. In line with the empirical work is the effort devoted to developing theoretical models that have proven successful in describing the phenomena observed in environments with social interactions, which can be categorized into three main classes. The local interactions models; for environments in which the population size is small such that the agents are aware of each others characteristics, or large population environments with random interactions. For example models of population games, learning and diffusion in social networks [Ellison, 1993, Young, 1993, Bala and Goyal, 1995, Vega-Redondo, 2007, Acemoglu et al., 2008] and provision of local public goods [Bramoullé and Kranton, 2007, Galeotti et al., 2010]. The global interactions model; in which the population size is infinitely large such that the agents only influence each other through the global average behavior of the population [Brock and Durlauf, 2001]. The third class generalizes the local and global interactions and is for situations in which the population size is infinitely large and the individual behavior is influenced by both the closest neighbors and the global distribution of strategies [Horst and Scheinkman, 2006].

The magnitude of social influence on individual behavior depends on the size of the interacting population and whether the interactions are localized within a bounded neighborhood or are global. If the size of the interacting population is large, we would expect that the cost of meeting or collecting information about all other agents becomes high both in terms of time and resources. So the best an agent would do is to take a statistical measure (for example a population average choice) that would give an idea of how the strategies are distributed across the population. In some social environments even when the interacting population is large individuals can still have small groups of other individuals that they would interact with on a regular basis, for example family members and relatives, classmates, co-workers, fellow researchers and many other forms of peer groups. If the sizes of these small groups or neighbors matters in determining the magnitude of social influence, and if the sizes of neighborhoods are heterogeneous across the population, then their distribution must in turn act to influence individual and aggregate behavior. The two other factors that determine the impact of social interaction on individual and aggregate behavior are the level of information available to the agents at the time of making decisions, and the nature of externalities derived from interacting with others. That is whether it is of strategic substitutes or complements, and either positive or negative externalities. For example in making a choice between competing technologies or softwares, if the game is of strategic complements such that individuals benefit from making compatible choices, then the larger the fraction of individuals making a compatible choice within ones neighborhood the higher the reward. Similarly in environments with positive externalities, such as provision of local public goods, an individual's decision to contribute will depend on the fraction of other individuals within her neighborhood that also contribute, moreover the marginal social cost within any neighborhood will be less than the marginal private cost.

In this paper, we study social interaction games with incomplete information in environments with large populations and in which individual decisions are affected by both local and global externalities. The local externality is limited within an agent's neighborhood, which we characterize by her ego-network[1], and the global externality is derived from aggregate distribution of strategies in a feedback manner. The agents will be identified with the sizes of the ego-network as their type; the size of ones ego-network is commonly referred to as the *degree* in social networks literature. The information is incomplete in the sense that agents have knowledge of the number of other agents they will potentially interact with or encounter (that will directly affect their decision) in the future but do not posses knowledge of their types or identity at the time of making a decision. Unlike in most of the previous models of social interactions where the sizes of neighborhoods is taken to be uniform across the population, our framework incorporates a heterogeneous distribution of neighborhood sizes. Since the population size is large, the agents posses only partial information about the distribution of sizes of ego-networks across the population such that they *dichotomously* assign probabilities from smallest possible size to the largest possible size. We borrow techniques from the literature of *complex networks* [Albert and Barabási, 2002, Newman, 2003] to address the problem of distribution of sizes of ego-networks. A network (of connections between agents) is said to be complex if its architecture does not exhibit uniform or clear patterns but rather displays substantial heterogeneities. The tools employed to study such interaction structures have their origin in random networks, formally initiated by Erdos and Renyi [1960]. When the network is complex, it is common to use probabilistic measures to capture its topological properties, such as the degree distribution and degree correlations. The agent's forecast of the distribution of neighborhood sizes across the population through dichotomous assignments therefore corresponds to the degree distribution. This concept has been applied to model local interactions in network games [Galeotti et al., 2010, Galeotti and Vega-Redondo, 2011].

Galeotti et al. [2010] study incomplete information games with local externalities. Agents posses partial information about the underlying interaction structure; they know the potential number of agents they will interact with but do not know the identity of their future partners. For example, in the case of choosing between competing technologies based on complementarity, the agent may know the potential number of other agents that she will interact with but does not know who these agents are. Other environments include the case in which a trader has knowledge of the potential number of customers but does not know their distribution relative to the number of other traders, or a Ph.D student deciding on the scientific theory to follow can anticipate the number of fellow researchers she will interact with in the future but does not know at the time of making the decision who these individuals are. By specifying the agents' neighborhoods in the social network as the source of their local externalities, Galeotti et al. [2010] characterize the impact of the level of information, the nature of the game and individual position within the network on the individual strategic behavior and payoffs. In a similar approach, Galeotti and Vega-Redondo [2011] study games with local externalities but with special focus on the continuous decision problem. Our framework differs in mainly two aspects. First, we consider a discrete choice case with both local and global externalities. By considering global interactions we are able to construct a micro-macro model and

[1] The words "neighborhood" and "ego-network" will be used interchangeably throughout the paper.

study the variations in aggregate characteristics as a result of variations in the interaction structure. Secondly, our framework is richer in the sense that we incorporate heterogeneity in individual preferences, which allows us to study other equilibrium phenomena; uniqueness and multiplicity.

A more closely related literature is the models of social interactions that employ the random fields approach of statistical mechanics [Blume, 1993, Durlauf, 1996]. The previous work in these models has considered either uniform neighborhood interactions and/or global interactions with an exception of Horst and Scheinkman [2006]. Horst and Scheinkman [2006] perform a qualitative analysis of games with local and global interactions with continuous decisions. By treating the interactions as an ergodic process, they are able to establish the existence of equilibrium, and identify conditions on the preferences and the extent of local interactions under which equilibrium outcome is unique. The main conditions are homogeneity in preferences and that the extent of local interactions should decay sufficiently fast with the distance between agents. Our framework differs in three main aspects: First, as already mentioned above, we consider discrete choice rather than continuous decisions, and that individual decisions are based on their expectations of others' actions not their *actual environment*; information is limited at the time of making decisions. Secondly, by considering an arbitrary interaction structure with specific neighborhood size distributions, a bound on the extent of local interactions is guaranteed and the upper-bound is determined by the structure that the neighborhood size distribution assumes. Third, we assume that individual actions are contingent on their type, and by assuming that the actions are symmetric in the argument (type of agent), homogeneity of preferences is guaranteed. Since our main concern in this paper is to address the effects of interactions structure on individual and aggregate behavior, we shall focus on the equilibrium outcome rather than the dynamics of the systems towards equilibrium. For this reason, we shall make the assumption of consistency between individual subjective and objective expectations at equilibrium as employed in Durlauf [1996], Brock and Durlauf [2001]. With this specification, we are able to characterize equilibrium outcomes; the existence, uniqueness and multiplicity, and by considering defined interaction structures we are able to obtain quantitative results on the effect of neighborhood size distribution on uniqueness and multiplicity of equilibria. We also explore the interaction between the parameters of the model; strength of local and global externalities.

In the remainder of the paper, we present the general framework in section 2, giving details on how the externalities depend on the interaction structure. We also present the general outline of the payoff structure and the characterization of equilibrium for the interaction system being studied. In section 3 we consider a special case of the binary action set, and explore the impact of changes in the interaction topology and the network parameters on the equilibrium properties. In section 4 we characterize the effects of local externalities on individual welfare ranking of the expected mean choice levels for the binary action set model developed in section 3. The conclusion and possible extensions are given in section 5

2 The model

2.1 Interaction structure and the local externality

We consider a social interaction game in which agents' neighborhoods are defined by their social networks or ego-networks, which is the number of other agents one directly interacts with. In this kind of setup, the interaction structure of the population can be defined by a graph $G(N, E)$, where N is the number of vertices and is equivalent to the number of agents (indexed $i = 1, \cdots, N$), and E is the set of edges linking pairs of agents. The neighborhood of an agent i is the set of agents directly linked to her. The the size of i's neighborhood is therefore equivalent to i's degree denoted by k_i, which will also denote the identity of or type for i. We are interested in determining the impact of the k's and their distribution across the population on individual and aggregate behavior given the model parameters. Since the population size is taken to be very large, we expect that the agents can posses only partial information about the distribution of k across the population. They can simply assign probabilities in a dichotomous manner. Borrowing from the literature of complex networks, the distribution of k will be defined by a probability density function $\mathbf{P}(k) = \{p(k)\}_{k=0}^{K}$, where $p(k)\ \forall k$ is the fraction of individuals with degree k [Newman, 2003, Vega-Redondo, 2007]. K is the maximum number of neighbors any agent can have. It follows from the configuration model that given $\mathbf{P}(k)$ the probability that any randomly chosen agent from the population is connected to a neighbor of degree k is defined[2] by the probability density function $\tilde{\mathbf{P}}(k) = \{\zeta(k)\}_{k=1}^{K}$, where

$$\zeta(k) = \frac{p(k)k}{\sum_{k'=1}^{K} p(k')k'} \tag{2.1}$$

The agents have limited information about their future partners; in particular they neither know their strategies nor their type, such that the interactions are random (not with a fixed predetermined set of neighbors). Since agents's actions are influenced by their social network, we can assume that actions are contingent on the size of the ego-network, that is $x(k)$. The implication is that an agent can make expectations about her future partners based on the partial information she possesses about the distribution of types across the population.

Denote by n_i as the neighborhood of i; a hypothetical set whose elements would be the potential future partners of i. What matters about the composition of n_i is its cardinality and the types of its elements. Denote by $x_{n_i} = \{\{x_j(k)\}_{k=1}^{K}\}_{j \in n_i}$ as the degree contingent strategy profile of n_i. We note that $x_{n_i} \in \mathbf{x}$, where $\mathbf{x} \in \mathbf{X}$ is the configuration of the population, or simply the population strategy profile, and $\mathbf{X}$ is the set of all possible configurations. We also note that the choice set X is discrete, of which we shall later in the paper focus on a binary set. We can define i's expectation of any one of her randomly chosen neighbor as follows

$$\mathbf{E}_i[x_j|\mathbf{P}] = \sum_{l=1}^{K} x_j(l)\zeta(l) \tag{2.2}$$

[2] This is normally referred to as the *excess degree* of a vertex, and is proportional to the number of "copies" of vertices with connectivity k within the population.

The expression for the expectations in (2.2) is for a single neighbor, but what we need is the conditional expectation(s) given that i has k_i neighbors, that is $\mathbf{E}_i[x_j|k_i,\mathbf{P}]$ $\forall j \in n(i)$. To achieve this, we make the use of multinomial distribution, defined as follows.

Denote by R_k, a rank vector that specifies the distribution of degrees of the neighbors of any randomly chosen agent with degree k. Since k is a $K \times 1$ discrete random vector, R_k is basically a support in k and defined as follows

$$R_k \equiv \left\{ r = (r_1, r_2, \cdots, r_K)^K : \sum_{l=1}^{K} r_l = k \right\} \tag{2.3}$$

such that for each $i \in N$ with degree k, $r = (r_1, r_2, \cdots, r_K) \in R_k$, for each $l = 1, 2, \cdots, K$, specifies the corresponding number of i's neighbors that have degree l, where $0 \leq r_l \leq k$. And it follows that $\sum_{l=1}^{K} r_l = k$. The distribution induced by each $r \in R_k$ follows a multinomial distribution given by:

$$P_k(r) = \frac{k!}{r_1! r_2! \ldots r_K!} \prod_{l=1}^{K} \zeta(l)^{r_l} \tag{2.4}$$

$P_k(r)$ can also be interpreted as the conditional (on k) degree distribution of i's neighbors; it captures the entire information i possesses regarding the degree of her neighbors. We can thus fully characterize the conditional expectation of i on her entire neighborhood as

$$\mathbf{E}_i[x_{n_i}(r)|k_i, \mathbf{P}] = \sum_{r \in R_k} P_k(r) \mathbf{v}_i(x_{n_i}(r)) \tag{2.5}$$

where $\mathbf{v}_i(x_{n_i}(r))$ is a function of the degree contingent choices of neighbors of i. For the sake of simple notations we denote $\mathbf{E}_i[x_{n_i}(r)|k_i,\mathbf{P}]$ by $\mathbf{E}_i[x_{n_i}]$. The form assumed by $\mathbf{v}_i(x_{n_i}(r))$ is determined by the nature of interactions; multiplicative, strategic complementarity or substitutes, public good games, and this in turn determines the rewards that an agent attains from interacting with her neighbors. To be more explicit, considering the definition of r above, by making an assumption that $x(k)$ is symmetric in k, $\mathbf{v}_i(x_{n_i}(r))$ can be written as follows:[3]

$$\mathbf{v}_i(x_{n_i}(r)) \equiv \mathbf{v}_i(\underbrace{x(1), \cdots, x(1)}_{r_1 \text{times}}, \underbrace{x(2), \cdots, x(2)}_{r_2 \text{times}}, \cdots, \underbrace{x(K), \cdots, x(K)}_{r_K \text{times}}) \tag{2.6}$$

Consider the following two cases; where the nature of interaction is such that there is a positive multiplicative externality, and that in which the externaility is a complementarity between i's action with the sum of the neighbors' actions (call this case the *positive summative externality*).

In the case of the positive multiplicative externality, the influence of j's action on i is taken to be the product of i's action and j's action, that is $x(k_i)x(k_j)$. Given k_i neighbors with multinomial vector r, we have $\mathbf{v}_i(x_{n_i}(r)) = x(1)^{r_1} x(2)^{r_2} \cdots x(K)^{r_K}$. Which is basically the Cobb-Douglas function. The resulting expressions of $\mathbf{E}_i[x_{n_i}]$ becomes

$$\mathbf{E}_i[x_{n_i}] = \sum_{r \in R_k} \left\{ \frac{k!}{r_1! r_2! \ldots r_K!} \prod_{l=1}^{K} [\zeta(l) x(l)]^{r_l} \right\}. \tag{2.7}$$

[3]Symmetry of $x(k)$ implies that all agents with degree k take the same action.

For the case of positive summative externality, $\mathbf{v}_i(x_{n_i}(r)) = \sum_{l=1}^{K} r_l x(l)$, such that

$$\mathbf{E}_i[x_{n_i}] = \sum_{r \in R_k} \left\{ \frac{k!}{r_1! r_2! ... r_K!} \sum_{l=1}^{K} \left(\prod_{l=1}^{K} [\zeta(l)]^{r_l} \right) r_l x(l) \right\} \tag{2.8}$$

It is possible to make alternative specifications on $\mathbf{v}_i(x_{n_i}(r))$ depending on the nature of interactions. In this paper we shall mainly adopt the positive summative externalities to illustrate the binary choice case considered in the sections to follow.

2.2 Empirical distribution

To take into account the global interactions, we introduce the empirical distribution, specifically the empirical average, as an additional parameter in the model. The empirical average associated with the action profile $\mathbf{x} = (x_1, x_2, \cdots, x_N)$ is defined as

$$\rho(x) = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{2.9}$$

Since the choices made depend on the interaction structure through the local interaction, the global average also endogenously depends on the interaction structure. The agent's expectation of the empirical average at the time of making a decision is $m_i(\rho) = \mathbf{E}\left[\rho(x)\right]$.

The definition in (2.9) sums the actions over the population. Though the most accurate would be to sum over the actions of all agents excluding i, we argue that since the population contains infinitely many agents, we expect that the contribution of each individual agent to the global average is negligible. So the difference between including i's action in computing her expectation of the empirical average and when it is excluded, is negligible. For this reason, we treat $m_i(\rho)$ to be uniform across agents, $m_i(\rho) = m \; \forall i \in N$.

2.3 Payoff structure

The utility, V_i, agent i derives from interacting with the entire population can then be expressed in an additive form as follows,

$$V_i(x_i, x_{-i}) = u_i(x_i) + L_i(x_i, x_{n(i)}) + G_i(x_i, x_{-i}) + \epsilon_i(x_i) \tag{2.10}$$

Where $u_i(.)$ is i's intrinsic (deterministic) utility for taking action x_i; it will specify how i evaluates choice x_i. $L_i(.)$ is the local interaction or externality[4]; the social utility derived from interaction with the neighbors. $G_i(.)$ is the global externality, which in this case will specify how an agent's choice is affected by the empirical distribution (or average). As an example, we can think of these structure of the utility in terms of the technology adoption, such that the intrinsic utility is individual preference among the available choices (without consideration of what other agents are choosing). $L_i(.)$ would be the benefits derived from compatibility of ones choice of technology with that of the co-workers for example, and $G_i(.)$ would be related

[4]The phrases "local and global interaction" will be used interchangeably with "local and global externality" through out the paper.

to the cost of acquisition, such that if more individuals acquire technology B rather than A, then the cost of B will be relatively lower.

$\epsilon_i(x_i)$ is i's individual random utility term; random in the sense that it involves the characteristics of an agent that are unobservable to the modeler, or in dynamic adjustment models of game theory it is related to the shocks on agent's periodic actions. We adopt an econometric interpretation for this model. Each agent $i \in N$ knows $\epsilon_i(x_i)$ at the time of her decision. ϵ_i's are assumed to be independent and identically distributed across agents and the distribution is common knowledge.

What we are interested in though is the expected utility of an agent given the available information; which include the number of potential neighbors, the probabilistic knowledge of the distribution of types (degrees) in the population. Let $U_i(.)$ denote the expected utility, then we have,

$$U_i(x_i, x_{-i}) = \mathbf{E}\left[V_i(x_i, x_{-i})\right] = u_i(x_i) + L_i(x_i, \mathbf{E}_i[x_{n(i)}]) + G_i(x_i, \mathbf{E}[x_{-i}]) + \epsilon_i(x_i) \tag{2.11}$$

where $\mathbf{E}_i[x_{n(i)}]$ and $\mathbf{E}_i[x_{-i}] \equiv m_i(\rho)$ are defined above.

The terms u, L and G are defined as follows; $u_i(x_i) = h_i x_i + c$, and for the sake of simplicity, in the analysis that follows h_i will be taken to be constant, that is $h_i = h$ for all $i \in N$.

We shall consider a particular environment in which social externalities are positive and exhibit strategic complementarity; for this special case, L and G will assume a quadratic form as follows.

$$\left.\begin{aligned} L_i(x_i, \mathbf{E}_i[x_{n_i}]) &= \alpha x_i \mathbf{E}_i[x_{n_i}] \\ G_i(x_i, \mathbf{E}[x_{-i}]) &= \beta x_i \mathbf{E}[x_{-i}] = \beta x_i m \end{aligned}\right\} \tag{2.12}$$

where α specifies the strength of i's expectation of her neighborhood on her marginal utility.[5] Similarly β is the strength of the global interaction.

2.4 Equilibrium characterization

To characterize the equilibrium outcomes in the environments with both local and global interactions, we consider two stages of the interaction process. In the first stage ("local equilibrium"), agents make decisions considering as given for each degree distribution and neighborhood, the empirical distribution (average). They then make a choice to maximize their utility given the anticipated choices of the neighborhood. That is, the action profile associated with the empirical average can be treated as a random variable, $f(m, k, \mathbf{P}) = \{f_i(m, k_i, \mathbf{P})\}_{i \in N}$, $f(m, k, \mathbf{P})$ is an equilibrium action profile if it satisfies

$$f_i(m, k_i, \mathbf{P}) \in \arg\max_{x_i \in X} U_i(x_i, \{f_j(m, k_j, \mathbf{P})\}_{j \neq i}, m, k, \mathbf{P}) \;\; \forall i \in N. \tag{2.13}$$

[5]We can also define α as a function of r, that is $\alpha(r) = \underbrace{\alpha'(1), \cdots, \alpha'(1)}_{r_1 \text{times}}, \cdots, \underbrace{\alpha'(K), \cdots, \alpha'(K)}_{r_K \text{times}}$.
Where α' is the strength of local interaction with a neighbor of a given degree.

In the second stage we consider the condition of self-consistency to define a "global equilibrium". That is, for an interacting system of infinitely many agents, the individuals' forecast of the average behavior (choice) coincides with the actual empirical average of the actions.

$$\mathbf{E}_i[\rho(x)] = \mathbf{E}\big[f(m,k,\mathbf{P})\big] \;\; \forall i \tag{2.14}$$

In the first stage we define the individual optimal strategies by a log-linear best response rule, which corresponds to the individual choice probabilities; that is the probability that an agent i takes action x_i conditional on the exogenous characteristics (degree, interaction topology, and other agents' strategies), $\text{Prob}(x_i|m,k_i,\mathbf{P})$.

By assuming that the behavior of each agent is independent of the behavior of the rest of the population (the agents play a non-cooperative game), we can characterize the conditional joint probability distribution of the choices across the population as

$$\text{Prob}(x_1,\cdots,x_N|m_1(\rho),\cdots,m_N(\rho),k_1,\cdots,k_N,\mathbf{P}(k)) = \prod_{i=1}^{N}\text{Prob}(x_i|m,k_i,\mathbf{P}(k)) \tag{2.15}$$

The equilibrium joint distribution in (2.15) above has been proven to correspond to the equilibrium (or rather long-run play) for the dynamic counterpart of the interaction process. In the dynamic case, agents play a non-cooperative game in which they are each given an opportunity to revise their strategy at random [Young, 1993, Blume, 1993].

From the equilibrium distribution in (2.15), we derive the corresponding equilibrium mean choice, and by assuming self-consistency of mean choice, this mean choice derived from the distribution is consistent with the mean choice m, that was anticipated by the agents at the time of making their decisions.

In what follows, we consider the special case of a discrete binary choice, $X = \{-1,1\}$, This enables us to obtain a relatively simplified result from which we derive a better insight on the role of interaction topologies in social interactions.

3 A binary choice case

We consider a special case with a binary action set, $X = \{-1,1\}$. Given their private utilities and their expectations about the choices of their future neighbors and the global average, the agents choose alternatives that optimize the expected utility U. That is i will choose 1 if $U_i(x_i = 1, x_{-i}) > U_i(x_i = -1, x_{-i})$, which after substituting (2.12) into (2.11) yields

$$\epsilon_i(1) - \epsilon_i(-1) > -2h_i - 2\alpha\mathbf{E}_i[x_{n_i}] - 2\beta m \tag{3.1}$$

This yields the conditional probability of choosing 1 to be

$$\text{Prob}(x_i = 1|m,k,\mathbf{P}) = 1 - \text{Prob}\left(\epsilon_i(1) - \epsilon_i(-1) \le -2h_i - 2\alpha\mathbf{E}_i[x_{n_i}] - 2\beta m\right) \tag{3.2}$$

By assuming that the $\epsilon(-1)$ and $\epsilon(1)$ are independent and extreme-value distributed (IID) across all agents and alternatives[6], it follows from McFadden [1984]

[6] If the ϵ's are IID, then

$$\text{Prob}(\epsilon(-1) - \epsilon(1) \le x) = \frac{1}{1 + exp(-\eta x)} \equiv \frac{\exp(\frac{\beta}{2}x)}{\exp(\frac{\beta}{2}x) + \exp(\frac{-\beta}{2}x)}$$

and Anderson et al. [1992] that

$$\text{Prob}(x_i = 1|m, k, \mathbf{P}) = \frac{\exp\left[\eta(2h + 2\alpha\mathbf{E}_i[x_{n_i}] + 2\beta m)\right]}{1 + \exp\left[\eta(2h + 2\alpha\mathbf{E}_i[x_{n_i}] + 2\beta m)\right]} \tag{3.3}$$

where η measures the impact of random component in the decision process,[7] the degree of dispersion in the random component of private utility. Put in another way, it is a measure of the inverse of heterogeneity of agents' preferences.

Alternatively, the choice probability can be expressed[8] in a general form as

$$\text{Prob}(x_i|m, k, \mathbf{P}) = \frac{\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]}{\sum_{x_i\in\{-1,1\}}\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]} \tag{3.5}$$

Since the ϵ's are IID across agents and alternatives, the joint conditional choice probability is equivalent to

$$\text{Prob}(x|m, k, \mathbf{P}) = \prod_i^N \frac{\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]}{\sum_{x_i\in\{-1,1\}}\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]} \tag{3.6}$$

Equation (3.6) can be seen as a product of the conditional probability distributions of N independent and identically distributed binary random variables taking on values $\{-1, 1\}$, with the probability given by (3.4). We can thus compute the expected value (objective expectation) for each of the $i \in N$ random variables as

$$\mathbf{E}[x_i|m, k, \mathbf{P}] = (-1)\text{Prob}(x_i = -1|m, k, \mathbf{P}) + (1)\text{Prob}(x_i = 1|m, k, \mathbf{P}), \tag{3.7}$$

which after making substitution[9] using (3.3) and a few steps of algebra yields

$$\mathbf{E}[x_i|m, k, \mathbf{P}] = \tanh\big(\eta(h + \alpha\mathbf{E}_i[x_{n_i}] + \beta m)\big) \tag{3.8}$$

Equation (3.8) is the conditional expected value of x_i. To obtain the expected value of x_i for all possible values of k, we integrate over degree probability density distribution (pdf). Denote the cumulative degree distribution (cdf) for the interaction structure

[7] A large η implies that the deterministic part of the utility function plays a vital role in the maximization process, while as η tends to zero, the error component of the utility dominates and the choice between $x = 1$ and $x = -1$ becomes a coin flip.

[8] The choice probabilities can also be expressed in the form

$$\begin{aligned}\text{Prob}(x_i|m, k, \mathbf{P}) &= \frac{\exp\left[\eta((hx_i + c) + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]}{\sum_{x_i\in\{-1,1\}}\exp\left[\eta((hx_i + c) + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]}\\ &= \frac{\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]\times\exp[\eta c]}{\sum_{x_i\in\{-1,1\}}\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]\times\exp[\eta c]}\\ &= \frac{\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m)\right]}{\sum_{x_i\in\{-1,1\}}\exp\left[\eta(hx_i + \alpha x_i\mathbf{E}_i[x_{n_i}] + \beta x_i m_i(\rho))\right]}\end{aligned} \tag{3.4}$$

[9] Substituting (3.3) into (3.7);

$$\mathbf{E}[x_i|m, k, \mathbf{P}] = (-1)\frac{\exp\left[\eta(-2h - 2\alpha\mathbf{E}_i[x_{n_i}] - 2\beta m)\right]}{1 + \exp\left[\eta(-2h - 2\alpha\mathbf{E}_i[x_{n_i}] - 2\beta m)\right]} + (1)\frac{\exp\left[\eta(2h + 2\alpha\mathbf{E}_i[x_{n_i}] + 2\beta m)\right]}{1 + \exp\left[\eta(2h + 2\alpha\mathbf{E}_i[x_{n_i}] + 2\beta m)\right]}$$

with degree distribution $p(k)$ by dF_p, then the expected value of x_i computed over all possible values of k becomes

$$\mathbf{E}[x_i] = \int \tanh\big(\eta(h + \alpha\mathbf{E}_i[x_{n_i}] + \beta m)\big)dF_p \tag{3.9}$$

Since k is a discrete random variable, and noting that the cdf can be expressed in terms of the pdf $dF_p = p(k)dk$, we can write the integral in equation 3.9 as a weighted sum of the *tanh*'s for all possible values of k, that is

$$\mathbf{E}[x_i] = \sum_{k=0}^{K} \tanh\big(\eta(h + \alpha\mathbf{E}_i[x_{n_i}] + \beta m)\big)p(k) \tag{3.10}$$

We postulate that if the empirical average defined by (2.9) exists, and is common to all agents, then from the argument of the law of large numbers,[10] this empirical average coincides with the expected value of the N random variables as outlined by (3.10).

From the law of large numbers,[11] for a given realization $\mathbf{x} = (x_1, \cdots, x_N)$, the sample mean choice will obey the following relation;

$$\lim_{N\to\infty} \rho_N \Rightarrow \mathbf{E}[x_i] \tag{3.11}$$

The self consistency condition implies that, at equilibrium, the realized empirical average $\rho_N \Rightarrow \mathbf{E}[x_i]$ for $N \to \infty$, is equivalent to the expected empirical average m. This condition leads to a closed equation whose root(s) are the steady state mean choice level(s) for a system of interacting agents with local and global interactions. That is,

$$m = \sum_{k=0}^{K} \tanh\big(\eta(h + \alpha\mathbf{E}_i[x_{n_i}] + \beta m)\big)p(k). \tag{3.12}$$

We would like to emphasize that in this type of analysis, like in Statistical mechanics, the interacting system of agents can be thought of as an *ensemble*; where m, the population average of the actions, is the state of the system, and represents the *macro-characteristic* of the system. The system can therefore be in any of the possible states (i.e set of values of m). The configuration of the system is determined by the microscopic characteristics, which are specifically speaking, agent's private preference, h, the neighborhood, k, and other parameters; η, α, and β. In the analysis, the point of focus is to study how variations at the microscopic level affects the state of the system (macro-characteristics).

[10] A definition from Ellis [1985, Appendix page 299]: Let $\{X_j; j = 1, 2, ...\}$ be a sequence of independent, identically distributed (i.i.d.) random vectors and define $S_n = \sum_{j=1}^{N} X_j$. If $\mathbf{E}\{\|X_1\|\}$ is finite, then $\frac{S_n}{N} \overset{a.s}{\to} \mathbf{E}\{X_1\}$

[11] It is important to notice that the law of large numbers does not apply to the agent's expectation of her neighborhood since the size of the neighborhood is bounded by K, a finite and small number of neighbors.

3.1 Equilibrium

We use the solution concept of *self consistent equilibrium*, whose main elements are defined in section 2.4, specifically (2.13) and (2.14). To be concise:

Definition 1 *A self consistent equilibrium is given by the population action configuration,* $\mathbf{x}^* = (x_1^*, \cdots, x_N^* |\ k, \mathbf{P})$ *that satisfies* (2.13), *with the corresponding set of values of* m^* *that solves* (3.12).

The next theorem will establish the existence of equilibrium, but first we state the property of *monotonicity* of individual actions, and its implications on the local externality.

Definition 2 *Actions or strategies are said to be monotone in degree if for strategic complements,* $k' > k$ *implies* $x(k') > x(k)$, *and* $x(k') < x(k)$ *for strategic substitutes.*

The property of monotonicity means that equilibrium actions are non-decreasing (non-increasing) in degree for the case of strategic complements (substitutes), and has a direct implication on the local externality in that a player with a higher degree will have a higher local externality than those with lower degree.

Consider a symmetric equilibrium in which all agents with the same degree k choose the same action, $x(k)$; for example if $k = 5$ then $x_i(5) = 1$ for all i with degree 5. Since $x(k)$ is monotone in k, for a choice set X and heterogeneous ego-networks $0 \leq k \leq K$, a threshold should exist above (below) which a certain choice is made.

Lemma 1 *Given that* $x_i(k)$ *is monotonically non-decreasing in* k, *for a binary set of* $x(k) = \{-1, 1\}$, *there exists a threshold* k_c *such that*

$$x(k) = \begin{cases} -1 & \text{if } k < k_c \\ 1 & \text{if } k \geq k_c \end{cases} \tag{3.13}$$

Lemma 1 is defined while assuming that alternative 1 is a better choice and hence there is a positive externality from choosing 1. The reverse could be used to define the case of negative externalities. It is also possible to define the case of multiple thresholds. For example in cases where there is a limit to which the size of the ego-networks can generate positive externality. That is for very small k an agent would prefer to choose -1, but for $k \geq k_c$ she would prefer to choose 1, till a limit where the size of k becomes expensive to maintain, above which -1 is preferable again. It also follows that if equilibrium can exist for the case of multiple threshold, then monotonicity of actions in degree is a sufficient but not necessary condition for existence of equilibrium configuration.

This characteristic of the decision process results from the fact that the game played is *decentralized* and is of *incomplete information*. The theorem follows.

Theorem 1 *For any given interaction structure* $P(k)$, *and equilibrium action configuration* x^*, *there exists at least one equilibrium point,* m^*, *that solves* (3.12).

Proof. Equation (3.12) can be rewritten in a functional form as,

$$f(m) = m - \sum_{k=0}^{K} \tanh\left(\eta(h + \alpha \mathbf{E}_i[x_{n_i}] + \beta m)\right) p(k). \tag{3.14}$$

We seek to show that there exist at least one value of m, m^*, for some constraints on the model parameters [12] and a given network structure, such that $f(m^*) = 0$.

The hyperbolic function $tanh(.)$ is monotone in its argument and is bounded on the interval (-1,1). The sum of monotone functions is monotonic, since k is bounded above by K, it follows that the weighted sum $W_s(k) = \sum_{k=0}^{K} \tanh\big(\eta(h + \alpha \mathbf{E}_i[x_{n_i}] + \beta m)\big)p(k)$, of $tanh(.)$, on the right hand side of (3.14) is monotonic and bounded. We also have that $m \in [-1,1]$, implying that $f(m)$ exist and is continuous in the interval $m \in [-1,1]$. It then follows from the intermediate value theorem [Foerster, 2004] that there exist at least one $m^* \in [-1,1]$ such that $f(m^*) = 0$.

For an explicit illustration consider the condition in which $h = 0$ and $\alpha = 0$, it can be easily checked that a root exists, $m^* = 0$, satisfying $f(m^*) = 0$. And the proof is true irrespective of either single or multiple thresholds of k.

■

3.2 Properties of Equilibrium

In the following analysis, we highlight the main properties of equilibrium outcome in terms of its uniqueness and multiplicity relative to the model parameters; specifically the parameter of agents' individual evaluation of choices, h, the parameter of heterogeneity, η and the strengths of the externalities α and β. We start with the baseline model as the one in which individual do not care about what the potential future neighbors will do but only care about the global share of agents taking a particular action, that is $p(k = 0) = 1$. We shall then use the *baseline model* for comparison with the case in which the network effects matter.

3.2.a Equilibrium for the baseline model, $p(k = 0) = 1$

To get a clear understanding of how the model parameters shape the equilibrium mean choice outcome, let us start by considering the simplest case in which agents' degrees are given by $\{k_i = 0\}_{i \in N}$, such that the agents are influenced by only the global externality. Denote $k = 0$ as k_0, and m for which $k = 0$ by m_0. For $p(k_0) = 1$ and $p(k) = 0$ for $k \neq 0$, equation (3.14) reduces to

$$m_0 = \tanh(\eta[h + \beta m_0]). \tag{3.15}$$

Under this specification, the agents enjoy only the global externalities on top of their private utility and the interaction system reduces to a global interaction model of Brock and Durlauf [2001], which has the following properties:

(i) When $h = 0$, and $\eta\beta < 0$, a unique root exist, $m_0^* = 0$; *a symmetric equilibrium.*

(ii) For $h = 0$ and $\eta\beta > 0$, three roots exist; one symmetric equilibrium and the two *asymmetric equilibria* that take on equal magnitude but opposite sign, m_-^* and m_+^*.

(iii) For $h \neq 0$ and $\eta\beta > 0$, there exists a threshold on ηh, h_c such that; when $|\eta h| < h_c$, multiple equilibria exist all of which are asymmetric, m_-^*, m_m^*, m_+^*, and when $|\eta h| > h_c$ the equilibrium is unique and takes on the sign of h.

[12] Model parameters include α, β, η and h

We briefly elaborate on the three properties mentioned above. Recall that η is a parameter that describes agents' heterogeneity in intrinsic taste, such that large η indicates a population in which agents share similar individual tastes, and a small η implies a large heterogeneity in tastes. Also recall that the agents individual judgement of the choices, h, can also be interpreted as the *private utility relative magnitude*[13] (or *bias*) of the two choices such that if $h > 0$ then an individual private judgement finds action 1 better or superior than -1. The behavior of the system is depicted in Figure 1.

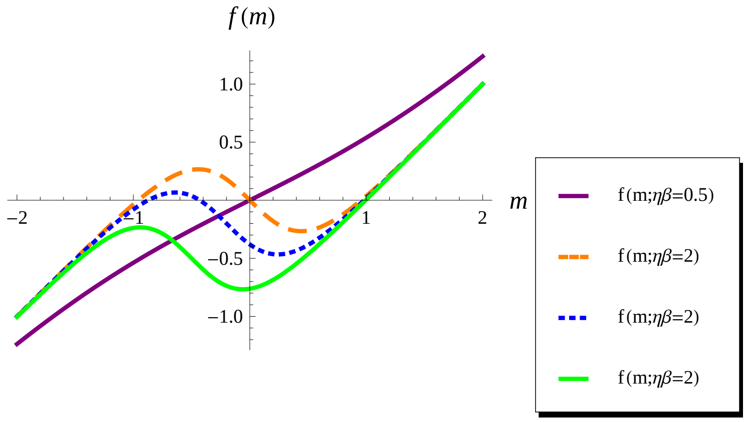

Figure 1 The equilibrium mean choice levels for the global interaction case. The bold purple line is for $\eta h = 0$ and $\eta\beta = 0.5$; dashed-orange line is for $\eta h = 0$ and $\eta\beta = 2$; dashed-blue for $\eta h = 0.4$ and $\eta\beta = 2$; and bold green corresponds to $\eta h = 1$ and $\eta\beta = 2$.

The first two properties above are for the case where individuals are indifferent between the two choices, that is $h = 0$. The first one basically says that if agents are indifferent between the choices, and if either there exists a large heterogeneity or a small strength of global influence or both, then the population will be split into one half adopting one choice and the other half adopting the other. We can also interpret the strength of global externality as a measure of the desire to conform to the global behavior, then property (i) says that when the desire to conform is low, agents base their decision on their individual evaluation of the choices, but since the agents are indifferent and heterogeneous then we have a half-half probability that each of them will choose one of the two alternative, hence $m^* = 0$.

The second property results from relatively higher levels of homogeneity in individual tastes and large strength of global externality. The magnitudes of m^*_- and m^*_+ are thus determined by the levels of homogeneity and strength of global interaction.

The third property emphasizes the case in which $h \neq 0$. When $|\eta h| < h_c$, we obtain multiple equilibria, and this represents a situation in which social influence can lead agents to choosing alternatives which are not necessarily ranked best by their individual evaluations. That is, let $h = 0.4$ such that individual private judgement finds action 1 better than -1, Figure 1 shows the possibility of obtaining an

[13] We define the relative magnitude as the utility difference between the two actions, that is, $h = \frac{1}{2}\{u_i(1) - u_i(-1)\}$

equilibrium mean choice level for which $m^* < 0$; the *co-existence* within the population of the superior choice together with the less superior but dominant choice. The phenomenon exhibited by social interaction that lead to the co-existence of competing alternatives has been used before to explain the co-existence of two competing technologies [Cowan and Cowan, 1998] and scientific theories [Brock and Durlauf, 1999].

3.2.b Equilibrium for $0 \leq p(k) \leq 1$

When $0 \leq p(k) \leq 1$, the equilibrium mean choice is shaped by the interaction topology, the strength and the size of local interactions; the size of agents' neighborhoods or ego-networks. Consider the marginal changes in the mean choice level with respect to the size of agents' neighborhood, and for a moment consider the case in which the size of neighborhoods is constant across the population, such that

$$f(m) = m - \tanh\Big(\eta\big[H(k) + \beta m\big]\Big) \tag{3.16}$$

where $H(k) = h + \alpha \mathbf{E}_i[x_{n_i}]$. The marginal change in mean choice level with respect to k is given by [14]

$$\frac{dm}{dk} = \frac{\alpha\eta\,\text{sech}^2\left[\eta\left(H(k)+\beta m\right)\right]}{1-\eta\beta\,\text{sech}^2\left[\eta\left(H(k)+\beta m\right)\right]}\frac{dH(k)}{dk} \tag{3.17}$$

A nonlinear relationship clearly shows up in (3.17), in which marginal change in mean choice level with respect to the size of the neighborhood depends on the current mean choice level and the model parameters. The right hand side of (3.17) is positive on condition that $dH(k)/dk > 0 \equiv d\mathbf{E}_i[x_{n_i}]/dk > 0$. Substituting for $\mathbf{E}_i[x_{n_i}]$ with the expression in (2.8) yields[15] $H(k) = h + \alpha k x(k)$ such that $dH(k)/dk = x(k) + k dx(k)/dk$. Since $x(k)$ is monotonically non-decreasing in k, it follows that $dH(k)/dk > 0$ and hence $dm/dk > 0$.

Consider the case in which $h > 0$, when compared to the baseline model for the same values of $|\eta h|$ and h_c, the threshold h_c required to attain a unique equilibrium for the case in which $\eta\beta > 1$ gets lower as k increases. That is if H_c is the threshold required for a unique equilibrium to exist, and let the mean choice level at that fixed point be m^*_u, then $H_c = h_c$ at $m^* = m^*_u$. It must therefore be that at $m^* = m^*_u$, $h < h_c$ in the presence of network externalities. The same argument can be followed to show that the level of individuals' evaluation or intrinsic preferences required for a specific alternative to be fully adopted by the population is lower in the presence of network externalities than in its absence.

When the neighborhood sizes are heterogeneously distributed across the population the situation gets a bit more complex, but never the less we shall explore

[14] The marginal change in the degree can be derived from $f(m)$ as

$$\frac{dm}{dk} = -\frac{df(m)}{dk}\frac{dm}{df(m)} = -\frac{df(m)}{dH(k)}\frac{dH(k)}{dk}\frac{dm}{df(m)}$$

[15] Denote the multinomial coefficient in (2.8) by $C(k)$. If $\{k_i = k\}_{\forall i \in N}$, the $C(k) = 1$, $p(k) = 1$, and $\sum_{l=0}^{K}\zeta(l)r_l x(l) = r_k x(k) \equiv kx(k)$

the cases for defined interaction topologies in the sections that will follow. Before embarking on that, the following corollary summarizes the above discussion and we also summarize the results on the stability of equilibria.

Corollary 1 *Let $x(k)$ be a degree contingent action set. If $x(k)$ is symmetric and monotonically non-decreasing in k, then the magnitude of the equilibrium mean choice level increases with the neighborhood size.*

3.3 Steady states stability to local adjustment

To check for the stability of the steady states mean choice levels[16] m^*_-, m^*_m, and m^*_+, we introduce the time variable t, such that the the mean choice at time t is denoted by m_t and similarly m_{t-1} for $t-1$. The dynamic process is such that expected choices and hence the corresponding mean choice at period t depends on the choices made at period $t-1$; this lead to the dynamic counter part of (3.12)

$$m_t = \sum_{k=0}^{K} \tanh\left(\eta(h + \alpha \mathbf{E}_i[x_{n_i}] + \beta m_{t-1})\right) p(k) \tag{3.18}$$

For a given interaction structure, the stability check can be carried out in the same procedure as in Brock and Durlauf [2001], we thus state the result here without reproducing the steps.

Proposition 1 *For a given interaction structure with degree distribution $P(k)$, and threshold k_c, it follows from proposition 4 of Brock and Durlauf [2001] that,*

(i) If equation (3.12) *exhibits a unique root, that root is locally stable.*

(ii) If equation (3.12) *exhibits three roots, then the steady state mean choice levels m^*_- and m^*_+ are locally stable whereas the steady state mean choice level m^*_m is locally unstable*

3.4 Equilibrium under arbitrary interaction topologies

This section looks at the effect of changes in the interaction topology on the equilibrium outcomes. The topology will be defined by an arbitrary degree distribution. Two cases will be considered; the Poisson network and the scale-free network degree distributions. First we characterize how the network parameters vary with the equilibrium mean choice level, then we shall consider the effect of varying the network structure. The questions that can be asked by these considerations are; What happens to the state mean if we increase the connectivity or degree density? What happens to the state mean if we increase or reduce the connectivity of one agent at the expense of others' connectivity? The first question can be answered by looking at the marginal changes in degree density for a given network, while the second can be answered by comparing the state equilibrium outcomes for different interaction topologies.

[16] m^*_- and m^*_+ are the two steady state mean choice levels associated with the case in which the largest parentage of agents choose -1 and 1 respectively, where as m^*_m is the steady state mean choice level associated with the case between m^*_- and m^*_+.

3.4.a Poisson degree distribution

Consider a situation in which the interaction topology assumes the Poisson distribution, that is,

$$p(k) = \frac{\exp(-z)z^k}{k!} \tag{3.19}$$

where $z = \sum_k kp(k)$ is the average degree distribution. We shall consider the positive externality described by (2.8). Denote the multinomial coefficient by $C(k)$ such that $C(k) = \frac{k!}{r_1!r_2!...r_K!}$. Substituting for $\zeta(k)$ (where k is equivalent to l) in (3.12) gives,

$$m = \sum_{k=0}^{K} \frac{e^{-z}z^k}{k!} \tanh\left(\eta\left[h + \alpha \sum_{r\in R_k}\left(C(k)\sum_{l=0}^{K}\prod_{l=1}^{K}\left(\frac{e^{-z}z^{l-1}}{(l-1)!}\right)^{r_l} r_l x(l)\right) + \beta m\right]\right) \tag{3.20}$$

We could follow the same steps as in subsection 3.2.b but the expression in (3.20) is cumbersome to handle analytically. So we resort to simulations in order to get a clear picture of the impact of network characteristics on the equilibrium mean choice level. We numerically simulate a Poisson degree distribution network as an approximation of Erdos and Renyi [1960] (hereafter ER network) for a large number of agents.

Figure 2 illustrates how the equilibrium mean choice, m^* varies with the density of the network. Setting the threshold parameter k_c at 4, we observe the behavior of m^* for values of $z \sim 2, 3, 5$, and the model parameters are set to be $\eta h = 0$, $\eta\alpha = 10$, and $\eta\beta = 1.5$. We set $\eta h = 0$ simply because we want to capture mainly the impact of the network topology, setting it otherwise does not change the result and most importantly the interpretations. The result in Figure 2 is consistent with that derived for a regular interaction topology in subsection 3.2.b. Increasing the degree density increases the size of ego-networks across the population. As z increases we observe a shift in equilibrium outcomes from a unique to multiple equiliria, and to a unique equilibrium again but with different signs as well as magnitude for the mean choice levels.

Proposition 2 *Let $x(k)$ be symmetric and monotonically non-decreasing in k, for a given k_c and model parameters, the equilibrium mean choice level m^*_+ increases with z and there exists a value of z above which the equilibrium is unique. And vise versa for m^*_-.*

Proof. Denote the fraction of agents from the neighborhood of an agent i with degree $k \leq k_c$ by $f_{k_c} = \sum_{k=1}^{k_c} p(k)$, and $f'_{k_c} = \sum_{k=k_c}^{K} p(k)$ to be the fraction of agents with degree $k > k_c$. The threshold property will imply that all agents that belong to f_{k_c} will choose alternative {-1}, and those in f'_{k_c} choose {1}. Increasing the density of connections within the population conditional on the degree of agent i (that is while keeping the degree, k, of agent i constant), increases f'_{k_c}. The effect of the increase of f'_{k_c} on i is through the choice probabilities[17], and in turn her conditional

[17] Choice probability of i is

$$\text{Prob}(x_i|m(\rho), k, \mathbf{P}) = \frac{\exp\left[\eta((hx_i) + \alpha\mathbf{E}_i[x_{n_i}] + \beta x_i m_i(\rho))\right]}{\sum_{x_i\in\{-1,1\}} \exp\left[\eta((hx_i) + \alpha\mathbf{E}_i[x_{n_i}] + \beta x_i m_i(\rho))\right]}$$

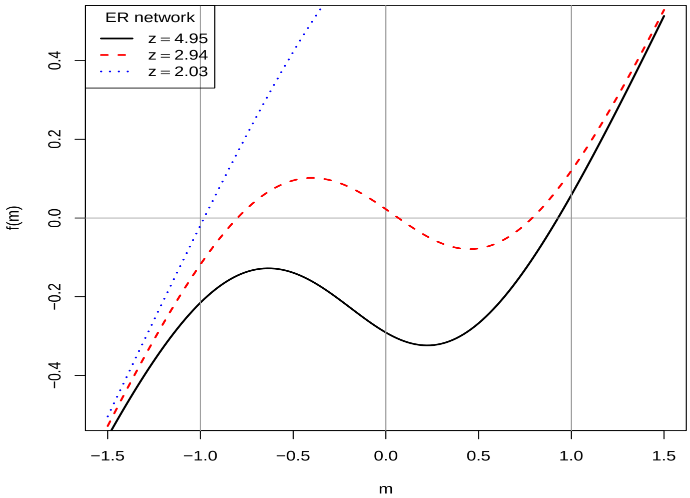

Figure 2   A plot of $f(m)$ for Poisson degree distribution for values of $k_c = 4$ and $z \sim 2, 3, 5$. The values of the model parameters are such that $\eta h = 0$, $\eta\alpha = 10$, and $\eta\beta = 1.5$.

expected choice. That is the larger f'_{k_c} the greater the probability of i choosing $\{1\}$. If this scenario applies to every agent in the population the result is a general shift of m^* to high magnitudes of m^*_+. A similar analysis and explanation holds if the threshold condition is reversed in such a way that the monotonicity of $x(k)$ is in the direction of {-1}. ■

3.4.b Scale-free degree distribution

The scale-free degree distribution assumes properties of the power law distributions. Specifically, the fraction of individuals in the network having degree k is described by the following expression

$$p(k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{k^{-\gamma}}{\mathcal{R}(\gamma)} & \text{if } k = 1, \cdots, \infty \end{cases} \tag{3.21}$$

where γ is a parameter and determines the decay of the distribution. $\mathcal{R}(\gamma) \equiv \sum_{k=1}^{\infty} k^{-\gamma}$ is the Riemann zeta function and has the property that it converges when $\gamma > 1$ and diverges for $\gamma \leq 1$. The average degree in this case is given by $z(\gamma) = \frac{\mathcal{R}(\gamma-1)}{\mathcal{R}(\gamma)}$. We notice that $z(\gamma)$ diverges when $\gamma < 2$ and converges to 1 when $\gamma \to \infty$. Some empirical studies place γ between 2 and 3 [Clauset et al., 2009, Barabasi and Albert, 1999]. Under scale-free degree distribution we thus have a situation where a great fraction of the agents posses a lower degree and few agents (hubs) posses very high degree. Scale-free degree distribution is thus the opposite representation of the Poisson degree distribution.

For a given k_c and model parameters, $\text{Prob}(x_i = 1|m(\rho), k, \mathbf{P})$ increases with z and $\text{Prob}(x_i = -1|m(\rho), k, \mathbf{P})$ decreases with z.

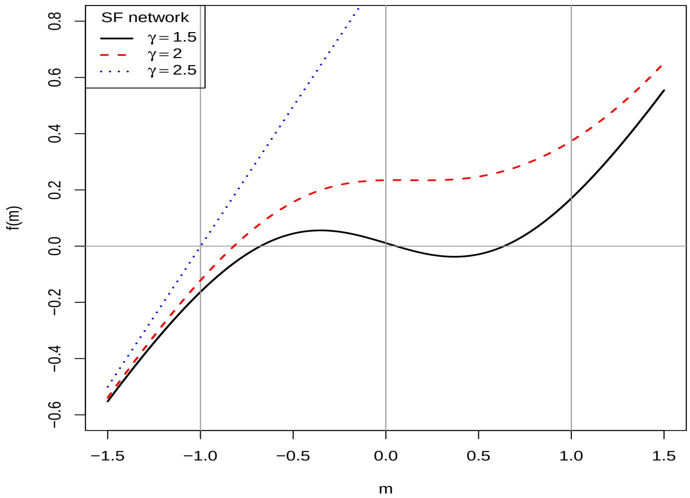

Figure 3 A plot of $f(m)$ for the scale free degree distribution. The network parameters are set at $k_c = 4$, $\gamma = 1.5, 2.0, 2.5$, and values of the model parameters are such that $\eta h = 0$, $\eta\alpha = 10$, and $\eta\beta = 1.2$.

Figure 3 shows the behavior of $f(m)$ for various network parameter (γ) values.[18] The individual evaluation parameter $h = 0$. For the same level of threshold $k_c = 4$, $f(m)$ is plotted for values of $\gamma = 1.5, 2.0, 2.5$. We observe that increasing γ has an opposite effect as to that of increasing the degree density in the Poisson degree distribution. Note that γ is a measure of the level of heterogeneity of the sizes of ego-networks across the population, such that lower values of γ imply a more heterogeneous population. It is observable from Figure 3 that the less heterogeneous the population in the distribution of ego-networks the more "stable" the population equilibrium state, and the reverse is true for a heterogeneous population. That is if the population state mean is initially at $m^* = m^*_-$, making a transition from $m^* = m^*_-$ to $m^* = m^*_+$ requires a small threshold level k_c, for $\gamma = 2.5$ than for $\gamma = 1.5$. This point will be clearer when we compare the two degree distributions in the next section. The above discussion is summarized in the following corollary.

Corollary 2 *Let $x(k)$ be symmetric and monotonically non-decreasing in k, for a given k_c and model parameters, the equilibrium mean choice level m^*_+ decreases with γ and there exists a value of γ above which the equilibrium is unique. The reverse is true for m^*_-.*

[18] Substituting for the value of $\zeta(l)$ for the scale-free network topology we have the state mean choice expressed in a functional form as follows

$$f(m) = m - \sum_{k=0}^{K} \frac{k^{-\gamma}}{\mathcal{R}(\gamma)} \tanh\left(\eta\left[h + \alpha \sum_{r \in R_k}\left(C(k)\sum_{l=0}^{K}\prod_{l=1}^{K}\left(\frac{l^{-(\gamma-1)}}{\mathcal{R}(\gamma-1)}\right)^{r_l} r_l x(l)\right) + \beta m\right]\right)$$

3.4.c Variation of threshold parameter k_c

We note that a similar variational effect on $f(m)$ can be produced by keeping the network parameters constant and vary the threshold parameter instead. Figure 4 shows the effect of varying k_c on $f(m)$, for Poisson distribution with $z = 5$. The adjustment on k_c produces a negative effect on the equilibrium mean choice level when compared to the adjustment on z; the effect on the shape of $f(m)$ is not exact to that produced by adjusting z but the same equilibrium points could be obtained. This has an important implication in real life decision problems with social interaction, which is that since the interaction topology is exogenously given, any social planner would not have to change the topology of interaction in the population to obtain a desired outcome of the mean choice level, but she could rather reduce or increase the threshold value through some form of *incentives*.

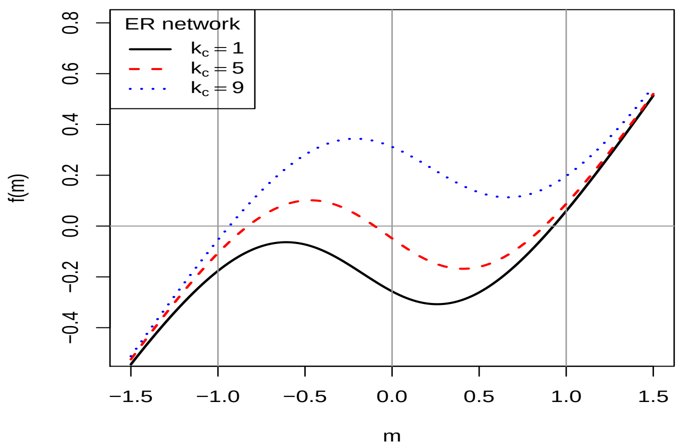

Figure 4 A plot of $f(m)$ for values of $k_c = 1, 5, 9$, with $z = 5$. The model parameters are set such that $\eta h = 0$, $\eta\alpha = 10$, and $\eta\beta = 1.5$. For intermediate values of k_c multiple equilibria exist, but for small and large k_c a unique m^* is obtained.

3.4.d Mean-preserving spread of $P(k)$

The question we ask here is how changing the interaction topology in a mean preserving manner would affect the equilibrium outcome; that is the equilibrium choice probabilities and in turn the equilibrium mean choice level. The Poisson and SF network degree distributions are generated for the same population size. The graphical representation is presented in Figure 5; $f(m)$ is plotted for $k_c = 4$ and similar conditions of the model parameters for both distributions. Two choices of γ are made such that $\gamma = 2.0$ produces a more heterogeneous SF degree distribution with the average degree $z \sim 3$ below that for Poisson network, and $\gamma = 1.5$ leads to a less heterogeneous population with $z \sim 10$.

Proposition 3 *Let $F(k)$ and $F'(k)$ respectively be the cumulative degree distributions for $P(k)$ and $P'(k)$. Also let $P'(k)$ be a mean-preserving spread of $P(k)$ such*

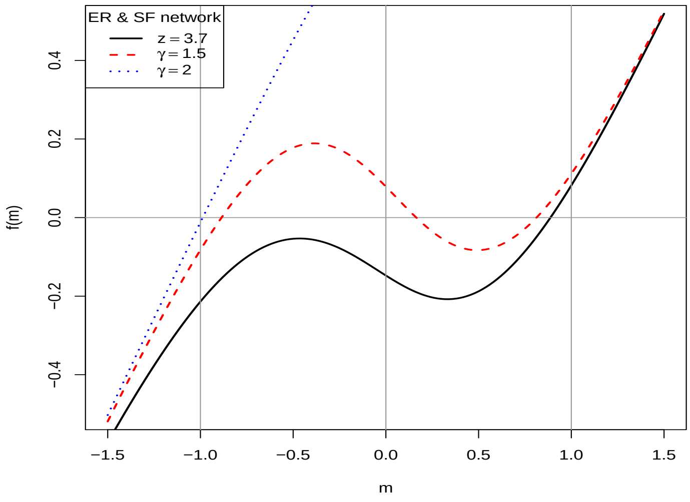

Figure 5 A plot of $f(m)$ for $k_c = 4$, $\eta h = 0$, $\eta\alpha = 10$, and $\eta\beta = 1.5$.

that there exists a $k = k_{eq}$ *for which* $F(k_{eq}) = F'(k_{eq})$*. If* $k_c < k_{eq}$*,* m^*_+ *under* $P(k)$ *is greater than* m^*_+ *under* $P'(k)$*, and vise versa for* $k_c > k_{eq}$*.*

Proof. Let $P(k)$ and $P'(k)$ respectively denote the degree density distributions for the Poisson and SF degree topologies, and $F(k)$ and $F'(k)$ be the respective cumulative degree distribution functions; the fraction of agents with degree less than or equal to k. Since both distributions are generated from the same population size, then $P'(k)$ for $\gamma = 2$ is close to being a mean preserving spread of $P(k)$. It follows that there must exist a value of $k = k_{eq}$, the "intersection degree", at which $F(k_{eq}) = F'(k_{eq})$. When $k < k_{eq}$, $F(k) < F'(k)$, which implies that if the threshold parameter $k_c < k_{eq}$, then $F(k_c) < F'(k_c)$. Since the fraction of agents choosing alternative $\{1\}$, m^*_+ is larger for $F(k_c)$ small[19], it implies that when $k_c < k_{eq}$, m^*_+ under $P(k)$ is greater than under $P'(k)$. In figure 6, when $\gamma = 1.5$ we have that $k_{eq} = 6$ and for $\gamma = 2$, $k_{eq} = 8$. On the other hand in figure 5 $k_c = 4$, such that $k_c < k_{eq}$, which proves the proposition. The reverse is also true for $k_c > k_{eq}$ ■

In Figure 6 there are two points for which $k = k_{eq}$, that is $k^l_{eq} = 2$ and $k^u_{eq} = 6$ for $\gamma = 1.6$, and $k^l_{eq} = 2$ and $k^u_{eq} = 8$ for $\gamma = 2$, where the superscripts l and u indicate lower and upper values of k_{eq} respectively. The two point of k_{eq} thus split the relationship between $F(k)$ and $F'(k)$ into three different regions; $k_c < k^l_{eq}$, $k^l_{eq} < k_c < k^u_{eq}$ and $k_c > k^u_{eq}$. In proposition 3 and Figure 5 we have considered $k^l_{eq} < k_c < k^u_{eq}$, where $F(k_c) < F'(k_c)$. Using the same argument, if $k_c < k^l_{eq}$ we have that $F(k_c) > F'(k_c)$, and proposition 3 implies that m^*_+ under $P'(k)$ is greater than m^*_+ under $P(k)$. Similarly if $k_c > k^u_{eq}$, we have that $F(k_c) > F'(k_c)$ and the result follows. Consider a situation in which -1 is the dominant choice in the population, and we (or a central planner in that matter) are interested in ensuring that 1 gets

[19] This relationship follows from the result in subsection 3.4.c

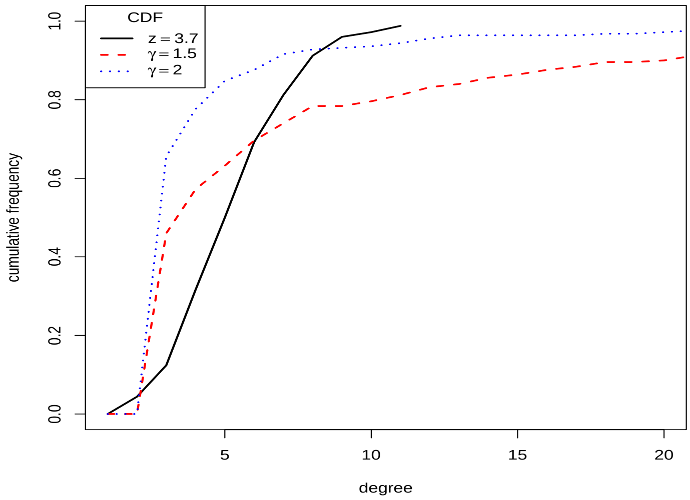

Figure 6   The cumulative degree distribution functions for Poisson degree distribution with $z = 3.6$ and SF degree distribution with $\gamma = 1.5, 2$.

adopted by a larger fraction of the population. One way of achieving the desired level of adoption of alternative 1 is by determining the existing interaction structure and consider adjusting k_c for the given model parameters, through incentives as highlighted in section 3.4.c above. The other alternative would be to take k_c and the model parameters as given and then determine the interaction topology that would best lead to the desired level of adoption for alternative 1. Proposition 3 then says that the scale-free degree (neighborhood sizes) distribution results to more individuals adopting 1 relative to Poisson degree distribution if and only if $k_c < k^l_{eq}$ and $k_c > k^u_{eq}$

3.5 Model parameters

In Figure 7 we plot the combination of values of the strengths of local and global interactions $\eta\alpha$ and $\eta\beta$ respectively, for which the equilibrium mean choice level is $m^* = 0.9$. That is the equilibrium outcome in which 90% of the population chooses any one of the two alternative. For example given two competing technologies, or products A and B, a policy maker or trader in that matter would like to estimate the combination of the strength of local and global externalities that would minimally lead to the greatest fraction of the population to adopt the desired among the two alternatives. The values of $\eta\alpha$ and $\eta\beta$ are generated for $k_c = 2$ in both the Poisson and SF degree distributions.

For a given level of heterogeneity in intrinsic preference η, the combinations of α and β vary between different interaction topologies even for similar conditions of k_c. Since the SF networks is "a prototype" of the *decentralized interaction structure* in which a few key figures, the hubs, play a central role of connecting the greater fraction of the population with lower degree, the result in Figure 7 highlights the conditions under which decentralized interaction structures may be more efficient in achieving full adoption when compared to other more uniform structures. We notice that for lower values of k_c the decentralized interaction structure may be more efficient than the random interaction topologies.

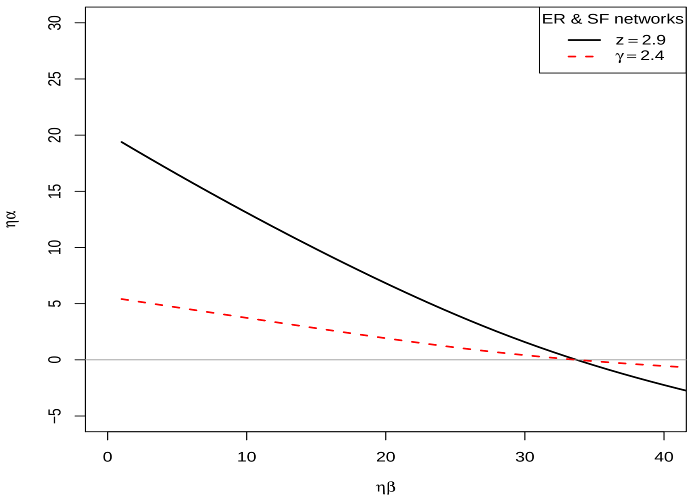

Figure 7 The combination of values of $\eta\alpha$ and $\eta\beta$ for which 90% adoption is attained. $k_c = 2$ and $\eta h = 0$.

4 Welfare rankings

We would like to establish for a given $P(k)$, which among the two stable equilibrium mean choice levels (m^*_- and m^*_+) will provide an agent with a higher level of expected utility conditional on the choices her neighbors make. Since the action are symmetric in degree, and the agents are typified by the degree, this is equivalent to computing the expected utility for a degree k. To achieve this, we compute the agent's expected utility prior to the realization of her random utility terms. We seek to characterize,

$$\mathbf{E}(\max_{x_i} U(x_i, x_{-i})|m^*_-, k) = \mathbf{E}[\max_{x_i}(hx_i + c + \alpha x_i \mathbf{E}_i[x^*_{n_i}] + \beta x_i m^*_- + \epsilon(x_i))] \quad (4.1)$$

and compare it to

$$\mathbf{E}(\max_{x_i} U(x_i, x_{-i})|m^*_+, k) = \mathbf{E}[\max_{x_i}(hx_i + c + \alpha x_i \mathbf{E}_i[x^*_{n_i}] + \beta x_i m^*_+ + \epsilon(x_i))] \quad (4.2)$$

where $x^*_{n_i}$ is the action profile of the neighbor's that would correspond to root m^*. Following from Small and Rosen [1981] and Anderson et al. [1992], equations (4.1) and (4.2) become

$$\mathbf{E}(\max_{x_i} U(x_i, x_{-i})|m^*_-, k)) = \eta^{-1}\Big(\ln\Big[\exp(\eta h + \eta c + \eta\alpha \mathbf{E}_i[x^*_{n_i}] + \eta\beta m^*_-) + \exp(-\eta h + \eta c - \eta\alpha \mathbf{E}_i[x^*_{n_i}] - \eta\beta m^*_-)\Big]\Big) \quad (4.3)$$

$$\mathbf{E}(\max_{x_i} U(x_i, x_{-i})|m_+^*, k)) = \eta^{-1}\Big(\ln \Big[\exp(\eta h + \eta c + \eta\alpha\mathbf{E}_i[x_{n_i}^*] + \eta\beta m_+^*) + \exp(-\eta h + \eta c - \eta\alpha\mathbf{E}_i[x_{n_i}^*] - \eta\beta m_+^*) \Big]\Big) \tag{4.4}$$

Consider first the case in which agents are indifferent between the two choices, such that $h = 0$, then $|m_+^*| > |m_-^*|$ only if $f_{k_c} < f_{k_c}'$, which yields $\mathbf{E}_i[x_{n_i}^*] > 0$. It follows from (4.3) and (4.4) that the expected utility under m_+^* is higher than that under m_-^*. The reverse is true for $f_{k_c} > f_{k_c}'$.

The case in which $h \neq 0$, we shall have four scenarios; the first is when $h > 0$ and $\mathbf{E}_i[x_{n_i}^*] > 0$, which will yield the expected utility under m_+^* to be higher than that under m_-^*. The second scenario is when $h < 0$ and $\mathbf{E}_i[x_{n_i}^*] < 0$; this yields the expected utility under m_-^* to be higher than that under m_+^*. The last two scenarios are symmetric, the one in which $h > 0$ and $\mathbf{E}_i[x_{n_i}^*] < 0$ and $h < 0$ and $\mathbf{E}_i[x_{n_i}^*] > 0$. If $h > 0$ and $\mathbf{E}_i[x_{n_i}^*] < 0$ but $h > \mathbf{E}_i[x_{n_i}^*]$, then m_+^* to be higher than that under m_-^*. The reverse is true for the opposite case and for the symmetric case. The result is summarized in the following proposition.

Proposition 4 *For a given interaction structure $P(k)$, the welfare rankings for any randomly chosen agent with degree k are as follows:*

(i) If $h = 0$ and $f_{k_c} < f_{k_c}'$, then the equilibrium associated with m_+^* yields a higher level of expected utility for every agent than the equilibrium associated with m_-^*. The reverse is true for $f_{k_c} > f_{k_c}'$.

(ii) If $h > 0$ and $f_{k_c} < f_{k_c}'$, then the equilibrium associated with m_+^* yields a higher level of expected utility for every agent than the equilibrium associated with m_-^*. The reverse is true for $h < 0$, and $f_{k_c} > f_{k_c}'$

(iii) When $h > 0$, $f_{k_c} > f_{k_c}'$ but $h > \mathbf{E}_i[x_{n_i}^*]$, then the equilibrium associated with m_+^* yields a higher level of expected utility for every agent than the equilibrium associated with m_-^*. The reverse is also true for $h < \mathbf{E}_i[x_{n_i}^*]$

5 Conclusion

Social interactions play a great role in many socioeconomic environments, and since the interactions are normally bounded within an individual's neighborhood, the sizes and the distribution of sizes of neighborhoods act to shape the individual and aggregate behavior. A large body of theoretical models have considered regular topologies, in which the sizes of neighborhoods is uniform across the population. Real world networks on the other hand indicate that social environments are governed by heterogeneous interaction topologies. In this paper, we defined the neighborhood at individual level, that is the size of their ego-network or the degree. We then consider the ego-networks to be heterogeneously distributed across the population, such that we model two types of heterogeneities, that due to individual private taste and that due to ego-networks. We consider two polar interaction structures, the Poisson degree distribution as an approximation of the random degree distribution for large number of agents, and the scale free degree distribution.

We characterize the existence of equilibrium in such systems and explore the influence of various interaction topologies on individual expectations and equilibrium aggregate outcome as measured by the mean choice level. We show that the topology of interaction greatly determines the multiplicity of equilibria as well as the magnitude of the equilibrium mean choice levels. The presence of network externalities can act to either reinforce the adoption of the superior alternative, or result in an equilibrium in which the inferior alternative remains the most dominant but coexisting with the superior alternative.

By considering a micro-macro approach, the model is very useful in predicting or at least deriving insights on the macroscopic parameter levels that can be obtained for a given interaction structure and distributions of microscopic characteristics. For this reason, we claim that the model is applicable for deriving insights for policy implications and optimal planning in general.

The model we have developed is general and can be applied to any interaction environments in which agents can be identified by types (not necessarily the degree) and that the individual strategies are contingent on the type. It can also be applied to cases where the interactions are governed by bipartite relations between different groups of actor, for example traders and consumers. Though we have considered only the case of positive externalities, single threshold, summative local externality and a binary choice set, several extensions in these directions are possible.

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