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**#2012-011**

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**UNU-MERIT Working Papers**

**ISSN 1871-9872**

**Maastricht Economic and social Research Institute on Innovation and Technology,  
UNU-MERIT**

**Maastricht Graduate School of Governance  
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# Do $Ak$ models really lack transitional dynamics?

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## Abstract

Contrary to a popular belief, the most popular  $Ak$  growth models display transitional dynamics once the representative agent and complete markets assumptions are overturned. The class of models is identified with diminishing-returns at individual but constant-returns at aggregate due to externality effects. Under incomplete markets, the former implies that dynasties with a lower levels of initial capital grow faster. This is picked up by the aggregate economy that passes through a long transitional period before it converges to its balanced growth path. During the transition period, aggregate consumption and output grow at the same rate but higher than that of capital.

*Key words:*

Transitional dynamics,  $Ak$  model, inequality dynamics, heterogeneous households, incomplete capital market

*JEL Classification:* D3, E, O4

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## 1. Introduction

In the last two decades, the  $Ak$  model has become one of the workhorse models of economic growth. Since its incorporation to the modern literature, (by the ingenious works of Romer, 1986, Rebelo, 1991 and Barro, 1990),<sup>1</sup> the model has been a quite important framework for understanding policy issues in long-run growth. Notwithstanding this popularity, the model is criticized for its lack of transitional dynamics (e.g., Mankiw, Romer and Weil, 1992). This worries growth economists as real economies feature somehow transitional dynamics shown by empirical evidence. The present paper argues that the most popular  $Ak$  models that follow Romer and Barro rather lack transitional dynamics only under the convenient but less realistic conditions of a representative firm or household, and a perfect capital market, however. With the relaxation of these assumptions, the models could display a rich transitional dynamics.<sup>2</sup>

The main property of a typical  $Ak$  growth model is the absence of diminishing returns to capital at the *aggregate* level. In an economy with the  $Ak$  technology, the marginal product of aggregate capital is constant at  $k$ , at all times. Therefore, the economy could display long-run growth without transitional dynamics. All aggregate variables – consumption, capital and

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<sup>1</sup>According to Aghion and Howitt (1998, p. 26), the  $Ak$  model is first introduced by Frankel (1962) although, Barro and Sala-i-Martin (2004, p. 63) think, the production function is first used by Neumann (1937).

<sup>2</sup>Angeletos and Panousi (2009) argue that the introduction of incomplete markets in neoclassical growth models upsets results that seem standard otherwise in regards to the macroeconomic effects of government spending.

output – could grow at the same rate. Two main reasons are often provided for the global absence of diminishing returns to capital: (i)  $k$ , in the  $Ak$  function, could represent a *broad* capital that includes both physical and human capital or, (ii) the production function at a firm level could be augmented by an economy-wide externality (such as a learning-by-doing technology of Arrow, 1962).<sup>3</sup> The latter is the basis of the Romer (1986, 1990) and Barro (1990) endogenous growth models, and many others after them.

This class of models in general could be characterized by a production function that features diminishing-returns at individual but constant-returns at aggregate. The production function at the firm level is a function of both private and social capital,<sup>4</sup> each facing diminishing returns. At the economy-wide level, however, capital escapes diminishing returns, which leads the models to generate endogenous growth. But, because this happens instantaneously, the models are not considered to display transitional dynamics unless attached to a certain alien assumption.<sup>5</sup>

The present paper shows that the class of  $Ak$  models displays transitional dynamics once the representative agent and complete markets assumptions are overturned, however. Under incomplete market, diminishing returns at

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<sup>3</sup>See Barro and Sala-i-Martin (2004, p. 63-66, 205-235) for a detailed discussion.

<sup>4</sup>The latter could represent a neighborhood effects or a public capital (e.g., Romer, 1986 and Barro, 1990, respectively).

<sup>5</sup>For instance, a habit formation in consumption (Carroll et. al., 1997, 2000, Gomez, 2008), a logistic population growth function (Guerrini, 2010), a vintage capital (Boucekkine et al. 2005), or a production function with asymptotically constant-returns to capital (Jones and Manuelli, 1990).

individual implies that dynasties with a lower levels of initial capital grow faster. Individual dynasties with different levels of initial capital experience different paths of capital, consumption and income growth. These bring a unique growth path of inequality in the economy. The dynamics of inequality is jointly determined with the dynamics of aggregate capital. Therefore, the economy passes through a transitional period of inequality and capital dynamics before it converges to its long-run balanced growth path. During the transition period, aggregate consumption and output grow at the same rate but higher than that of capital.

At individual, consumption and output grow at the same rate but *different* from that of capital. For some dynasties, capital grows faster than consumption, and conversely. Households with below-average capital experience a higher growth rate of capital than consumption. In the steady-state, the growth rates of individual variables – individual consumption, income and capital – converge, and, are equal to the long-run growth rate of the economy.

The paper relates to the endogenous growth literature that provides different ways of recovering transitional dynamics in  $Ak$  models (e.g., Tamura, 1991, Acemoglu and Ventura, 2002).<sup>6</sup> Tamura (1991) displays transitional dynamics in  $Ak$  model with a neighborhood effects of human capital but in a two-sector setting. He models the externality effects in the human cap-

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<sup>6</sup>See also Footnote 5.

ital accumulation sector whereas he applies an  $Ak$  technology in the final goods sector. The model generates transitional dynamics at individual but at aggregate level. Acemoglu and Ventura (2002) model individual countries (vis-à-vis the world economy) where each possesses  $Ak$  technology but shows transitional dynamics when international trade and changes in the terms of trade change the rate of return of capital in the short-run. In contrast, we show here that a one-sector  $Ak$  economy displays transitional dynamics at both individual and aggregate levels in an environment with heterogeneous agents and imperfect credit market.

The paper may also relate to the literature of imperfect credit markets, inequality and growth (e.g., Loury, 1981, Galor and Zeira, 1993, Aghion et al., 1999, Benabou 1996, 2000, 2002), especially, with respect to linking inequality dynamics to the dynamics of aggregate capital. But, this literature does not focus on transitional dynamics in  $Ak$  models.

The rest of the paper proceeds as follows. Section 2 sets up a growth model, which is  $Ak$  at aggregate as the production function at a firm level is augmented by an economy-wide externality. Section 3 discusses two cases, – a representative agent and a complete market economy–, where the  $Ak$  model lacks transitional dynamics. Section 4 relaxes these assumptions to derive transitional dynamics in the  $Ak$  model. Section 5 looks into the model’s closed-form solution to reveal various interesting transitional properties of the  $Ak$  model at both individual and aggregate levels. Section 6 concludes.

## 2. The Model

### 2.1. Preference and household optimization

Suppose a continuum of infinitely-lived heterogeneous households. The  $i$ th household initially endowed with  $k_0^i$  units of capital and a unit of inelastic labour. Each household operates a privately-owned-firm.<sup>7</sup>

The  $i$ th dynasty maximizes her utility in accordance to the utility function:

$$U(c_t^i) = \sum_{t=0}^{\infty} \beta^t (c_t^{i1-\delta} - 1) / (1 - \delta) \quad (1)$$

subject to the budget constraint,

$$c_t^i + k_{t+1}^i = (1 - \tau) y_t^i \quad (2)$$

where  $c_t^i$ ,  $k_{t+1}^i$  and  $(1 - \tau) y_t^i$  are the  $i$ 's consumption, saving and after tax income, respectively.<sup>8</sup> The first order conditions associated to the Lagrangian ( $\mathbf{L}$ ),

$$\mathbf{L} = (c_t^{i1-\delta} - 1) / (1 - \delta) + \lambda_t^i ((1 - \tau) y_t^i - c_t^i - k_{t+1}^i) \quad (3)$$

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<sup>7</sup>This type of individual entrepreneurship is not uncommon in the literature (see, for instance, Benabou, 2002 and Angeletos and Calvet, 2006).

<sup>8</sup>Variables with(out) superscript  $i$  represent individual (aggregate) values.



of the optimization problem are

$$-\lambda_t^i + c_t^{-\delta} = 0 \quad (4)$$

$$\lambda_{t+1}^i \beta (1 - \tau) \partial y_{t+1}^i / \partial k_{t+1}^i - \lambda_t^i = 0 \quad (5)$$

From (4) and (5), the Euler equation is given by,

$$c_{t+1}^i / c_t^i = \beta^{1/\delta} \left( (1 - \tau) \partial y_{t+1}^i / \partial k_{t+1}^i \right)^{1/\delta} \quad (6)$$

together with the transversality condition:

$$\lim_{t \rightarrow \infty} \beta^t \lambda_t^i k_{t+1}^i = 0 \quad (7)$$

## 2.2. Production function

The firm of agent  $i$  has the following constant return to scale production function,

$$y_t^i = k_t f(\varphi_t^i) \quad (8)$$

$$\varphi_t^i \equiv k_t^i / k_t; \varphi_t = 1; k_t \equiv \int_i k_t^i \quad (9)$$

$$y_t = A_t k_t; A_t(\cdot) \equiv \int_i f(\varphi_t^i) \quad (10)$$

where  $k_t^i$  and  $y_t^i$  are the firm's capital and output, respectively.

There exists diminishing-returns to factors, at a firm level,

$$f'_{\varphi_t^i}(\varphi_t^i) > 0; f''_{\varphi_t^i}(\varphi_t^i) < 0 \quad (11)$$

However, at aggregate, constant-returns to capital applies, *ceteris paribus*. This implies  $A_t(\cdot)$  is not a function of  $k_t$ . The marginal product of aggregate capital does not vary with changes in  $k_t$  at all times. These could be the reasons behind  $Ak$  models' capability of generating endogenous growth but also their "lack" of transitional dynamics.

### 3. Under what circumstances the $Ak$ model lacks transitional dynamics?

The model described above will have no transitional dynamics if there is (1) a representative agent and/or (2) a perfect credit market. The intuition is straightforward: In both cases, the rental price of capital is fixed. This leads the economy to grow at a constant rate at all times, in both the short- and long-run.

We will first discuss each of these scenarios in more detail before we proceed to the next section where we relax the assumptions to recover transitional dynamics in the model.

#### **Case 1.** *A representative agent model*

Suppose a representative firm but heterogeneous households.<sup>9</sup> Then, the production function of the firm is given by, considering (8),

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<sup>9</sup>According to Caselli and Ventura (2000), the representative agent assumption does not necessarily rule out households' heterogeneity.

$$y_t = f(1)k_t \equiv Ak_t \quad (12)$$

where  $A$  is simply a constant. The representative firm maximizes profit when the marginal product of capital is equal to the rental price of capital ( $r_t$ ),

$$\partial y_t / \partial k_t = r_t = A \quad (13)$$

The budget constraint that the  $i$ th household faces is given by

$$c_t^i + k_{t+1}^i = (1 - \tau)(1 + r_t)k_t^i \quad (14)$$

Thus, the Euler equation for the optimization of the  $i$ 's individual utility (1) subject to the budget constraint (14) is

$$c_{t+1}^i / c_t^i = (\beta(1 - \tau)(1 + r_t))^{1/\delta} \quad (15)$$

From (13),  $r_{t+1} = r_t = r$ .  $r_t$  does not vary with changes in  $k_t$ , which is the characteristic of a typical  $Ak$  model. Therefore, the growth rate of aggregate consumption is constant:

$$\gamma_t^r + 1 \equiv c_{t+1}/c_t = (\beta(1 - \tau)(1 + r))^{1/\delta} \quad (16)$$

In Barro and Sala-i-Martin (2004), it is shown that  $\gamma_t^r + 1 = c_{t+1}/c_t = k_{t+1}/k_t = y_{t+1}/y_t$  during both the transition period and the steady-state, using the transversality condition. Moreover, the consumption-capital ratio

$(c_t/k_t)$  is constant at any point in time.

**Case 2.** *A perfect credit market*

Resume the assumption that both firms and households are heterogeneous. The firms are owned by members of the households. But individual households are allowed to borrow and lend with a market interest rate of  $r_t$ . The government imposes a flat rate tax ( $\tau$ ) on all incomes.<sup>10</sup> Then, the budget constraint that the  $i$ th household faces is given by

$$c_t^i + k_{t+1}^i + b_{t+1}^i = (1 - \tau) (y_t^i + b_t^i (1 + r_t)) \quad (17)$$

where  $k_{t+1}^i$  and  $b_{t+1}^i$  represent a portfolio of capital and bond, respectively.

The first order conditions of the Lagrangian,

$$\mathbf{L} = (c_t^{i1-\delta} - 1) / (1 - \delta) + \lambda_t^i [(1 - \tau) (y_t^i + b_t^i (1 + r_t)) - c_t^i - k_{t+1}^i - b_{t+1}^i] \quad (18)$$

of the optimization problem, are given by eqs. (4), (5) and,

$$\lambda_{t+1}^i \beta (1 - \tau) (1 + r_{t+1}) - \lambda_t^i = 0 \quad (19)$$

Then, the no-arbitrage equilibrium condition is, from (5) and (19),

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<sup>10</sup>Applying a different tax system may distort prices but will not change the essence of the basic argument.

$$1 + r_{t+1} = \partial y_{t+1}^i / \partial k_{t+1}^i \tag{20}$$

Therefore, each individual trades capital until her marginal rate of return to capital equals the market interest rate of capital:  $1 + r_{t+1} = \partial y_{t+1}^i / \partial k_{t+1}^i = \partial y_{t+1}^j / \partial k_{t+1}^j$  for two individuals  $i$  and  $j$ , respectively. This implies each individual invests the same amount of capital. Therefore, their next period income is identical.<sup>11</sup> Substituting (20) into the Euler equation (6), we will arrive in a similar conditions to (15) as  $r_{t+2} = r_{t+1} = r$ . Initial inequality vanishes instantaneously and the model does not feature any transitional dynamics. It turns into a version of the representative agent type economy where a representative household owns a representative firm.

#### 4. Transitional dynamics in $Ak$ models

The lack of transitional dynamics in the above models is mainly due to the lack of movements in the rate of return to capital. The presence of a perfect capital market creates an instantaneous equalization of intra- and inter-temporal individual households' productivity in the economy. This fixes the rate of return to capital and hence leads the economy to converge to its long-run equilibrium path without transition. Introducing imperfection in the capital markets, however, could cause initial individuals' productivity

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<sup>11</sup>Beanbou (1996) provides similar argument but with intra-temporal household allocation.

differences to persist, which effects inequality persistence and, consequently, transitional dynamics in the economy.

When the credit market is imperfect, individuals' investment opportunity will be limited to the resource they have in hand. When this is coupled with diminishing returns to capital, it leads to a persistence of intra- and inter-generational inequality. The dynamics of inequality is jointly determined with the dynamics of aggregate capital. Thus, the growth rates of the economy, aggregate capital and consumption are not constant during transition but evolve with inequality at different rates until the point that the economy converges to its long-run growth path.

#### 4.1. *Heterogeneous agents with incomplete capital markets*

Suppose now that households are heterogeneous in terms of their endowment  $k_t^i$ , and the credit market is imperfect. Also, initial wealth and income are lognormally distributed, such as  $\ln k_0^i \sim N(\mu_0, \sigma_{0,k}^2)$  and, hence,  $\ln \varphi_0^i \sim N(0, \sigma_{0,k}^2)$ . In this case, the marginal product of aggregate capital ( $k_t$ ) is determined by the level of inequality ( $\sigma_{t,k}^2$ ):

**Lemma 1.** *Given (8), (11), and individual heterogeneity in  $k_t^i$ ,*

$$\int_i [f(\varphi_t^i)] = \mathbb{E}[f(\varphi_t^i)] = g(\sigma_{t,k}^2, \cdot) \quad (21)$$

where the dot ( $\cdot$ ) here represents some parameters.

**Proof.** The production function is  $Ak$  at aggregate means that the marginal product of aggregate capital ( $\partial y_t / \partial k_t$ ) does not change with  $k_t$ . From the

Jensen's inequality, one sees that

$$\mathbb{E} [f (\varphi_t^i)] \leq f (\mathbb{E} [\varphi_t^i]) = f (1) \quad (22)$$

The inequality in (22) holds iff there is no wealth/income inequality,  $\sigma_{t,k}^2 = 0$ .<sup>12</sup> Thus,  $\mathbb{E} [f (\varphi_t^i)]$  is a function of inequality ( $\sigma_{t,k}^2$ ) and some parameters associated to the function. ■

Note also that,  $\mathbb{E} [f'_\varphi (\varphi_t^i)]$  is constant at  $k_t$  and  $\mathbb{E} [f'_\varphi (\varphi_t^i)] \geq f'_\varphi (\mathbb{E} [\varphi_t^i]) = f'_\varphi (1)$ , considering (11). Thus, we can write

$$\mathbb{E} [f'_\varphi (\varphi_t^i)] = h (\sigma_{t,k}^2, \cdot) \quad (23)$$

Similarly,  $\mathbb{E} [\ln f (\varphi_t^i)]$  and  $\mathbb{E} [\ln f'_\varphi (\varphi_t^i)]$  can be defined in terms of  $\sigma_{t,k}^2$ .

#### 4.1.1. Aggregation and distribution

The Appendix derives the distributional dynamics of capital,

$$\sigma_{t+1,k}^2 = l (\sigma_{t,k}^2, \cdot) \quad (24)$$

and the aggregate Euler equation, under the assumptions that households are heterogeneous and the capital markets are imperfect,

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<sup>12</sup>We show that later on this is the case at equilibrium.

$$c_{t+1} = c_t (\beta (1 - \tau))^{\delta^{-1}} \exp \{ 0.5l (\sigma_{t+1,k}^2, \cdot) - 0.5\sigma_{t+1,k}^2 + \delta^{-1}s (\sigma_{t+1,k}^2, \cdot) \} \quad (25)$$

Eq. (25) defines the growth rate of aggregate consumption as a function of the distributional dynamics of capital ( $\sigma_{t+1,k}^2$ ). Aggregating the budget constraint (2) is straightforward, considering Lemma 1,

$$k_{t+1} = g (\sigma_{t,k}^2, \cdot) (1 - \tau) k_t - c_t \quad (26)$$

First, note that, the essence of the current model lies in eq. (24). If the dynamics of inequality vanishes, the model behaves as the textbook  $Ak$  model, which is exactly what happens in the steady-state.

**Definition 1.** *A steady state in the  $Ak$  growth model is a balanced growth path where (i) aggregate consumption, capital, and output grow at a constant rate,  $\gamma + 1 \equiv c_{t+1}/c_t = y_{t+1}/y_t = k_{t+1}/k_t$  and, (ii) the dynamics of inequality converges to a constant distributional level:  $\sigma_{t+1,k}^2 = \sigma_{t,k}^2 = \sigma^2$ .*

**Proposition 1.** *In the steady state, the inequality dynamics in (24) converges to a unique stable steady-state where there is no inequality:  $\sigma_{t+1,k}^2 = \sigma^2 = 0$ .*

**Proof.** Suppose on the balanced growth path where  $\ln (c_{t+1}/c_t)$  is constant,  $\sigma_{t+1,k}^2 \neq 0$ . The latter implies that there is at least one individual  $l$  who has a larger capital than that of the average ( $k_{t+1} < k_{t+1}^l$ ). Thus, from (6) and (11),  $\ln (c_{t+1}/c_t) > \ln (c_{t+1}^l/c_t^l)$ . This catching-up continues though at a *declining* rates as the wealth gap closes. Individuals with low levels of capital



grow faster due to diminishing returns. But, this contradicts  $\ln(c_{t+1}/c_t)$  is constant on the balanced growth path; therefore,  $\sigma_{t+1,k}^2 = \sigma^2 = 0$ .

Eqs. (6) and (11) also ensures the stability of the inequality dynamics. Suppose the economy is in a steady-state distribution where  $\sigma_0^2 = 0$ , initially. But, during the next period, it appears that an individual posses capital a little bit higher than the rest of the population,  $\sigma_1^2 \neq 0$ . Then, according to (6) and (11), in the following periods ( $t = 1, 2, 3, \dots$ ), the individual's capital grows slower than everybody else until the point her capital converges to the rest of the population. ■

Second, from (25), it is obvious that the growth rate of consumption,  $\ln(c_{t+1}/c_t)$ , is not constant as long as  $\sigma_t^2$  keeps varying. The latter dictates the distribution dynamics and, hence, the dynamics of aggregate capital, output and consumption during the transition period. We will thus have the following proposition:

**Proposition 2.** *The Ak model with heterogeneous households and incomplete capital market displays transitional dynamics.*

Figure 1 depicts the phase diagram related to (24) and (25). It demonstrates the transitional dynamics with respect to aggregate consumption. The vertical axis represents the equilibrium growth rate of consumption where  $\sigma^2 = 0$ . The leftward-pointed arrows indicate the decline in inequality dynamics while the upward-pointed arrows show the  $Ak$  economy is a growing economy. It approaches its long-run growth path, as the inequality declines. Eventually,  $\sigma_t^2$  converges to zero and, hence, the model behaves

similar to the textbook  $AK$  model.

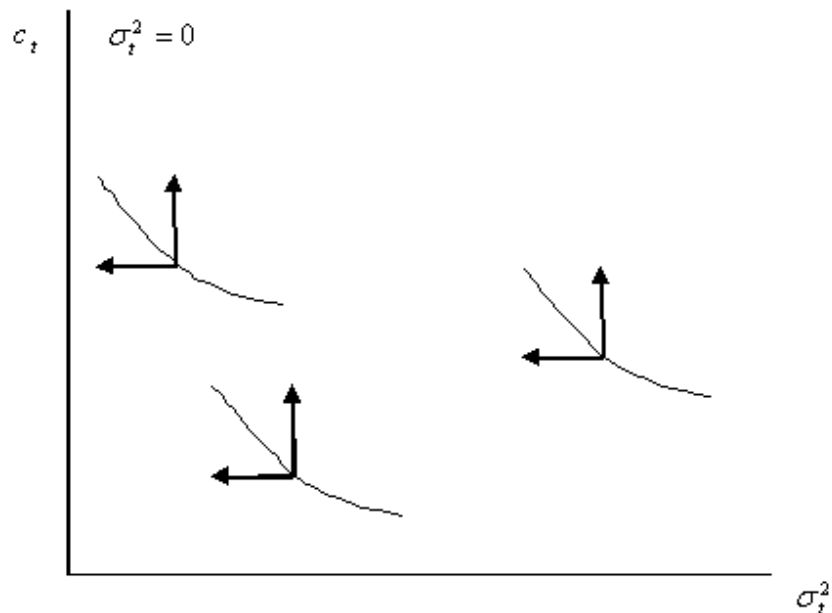


Figure 1: Transitional dynamics in  $Ak$  models with heterogeneous households and imperfect capital market. The left-pointed arrows indicate a declining in inequality dynamics as the economy continues to grow, as shown by the upward-pointed arrows.

## 5. Closed-form solution

This section further examines the  $Ak$  model by introducing a closed-form solution. An interesting aspect of the model is that it features different properties at individual and aggregate levels, during the transition period. Individual dynasties with different levels of initial capital follow different paths of capital, consumption and income growth. The growth rates of an individual's consumption and income are always the same but different from that of capital. The economy, on the other hand, shows a unique paths of

inequality and growth of aggregate variables. Aggregate consumption and output grow at the same rate but higher than that of aggregate capital.

### 5.1. Cobb-Douglas technology and logarithm preference

When the production function (8) is Cobb-Douglas,

$$y_t^i = k_t f(\varphi_t^i) = k_t (k_t^i/k_t)^\alpha \quad (27)$$

and  $\delta = 1$  in (1), the Euler equation is given by,

$$\gamma_{t+1,c}^i + 1 \equiv c_{t+1}^i/c_t^i = \beta\alpha(1-\tau) (k_{t+1}^i/k_{t+1})^{\alpha-1} \quad (28)$$

Then, using standard methods, we can easily obtain analytical solutions for the model:

$$k_{t+1}^i = \beta\alpha(1-\tau) k_t (k_t^i/k_t)^\alpha \quad (29)$$

$$c_t^i = (1-\beta\alpha)(1-\tau) k_t (k_t^i/k_t)^\alpha \quad (30)$$

Aggregating (29) yields the dynamics of aggregate capital investment:

$$\gamma_{t+1,k} + 1 \equiv k_{t+1}/k_t = \beta\alpha(1-\tau) \exp\{\alpha(\alpha-1)0.5\sigma_{t,k}^2\} \quad (31)$$

whereas taking the log and then the variance of (29) yield the distributional dynamics associated to capital investment,

$$\sigma_{t+1,k}^2 = \alpha^2 \sigma_{t,k}^2 \quad (32)$$

### 5.2. Individual dynamics

Eqs. (28) and (29), together with (31) and (32), characterize the dynamics at individual. The dynamics of aggregate capital and inequality thus determine the evolutions of individual consumption and capital growth. The intuition lies in the presence of externality effects in the economy. The effect of inequality on individual households is channeled through its effect on aggregate capital (31) whereas the latter has a direct impact on individual households' productivity through the externality effects (27).

From (27), (29) and (30), individual capital investment ( $k_{t+1}^i$ ), consumption ( $c_t^i$ ) and income ( $y_t^i$ ) grow at the same rate at all times:

$$k_{t+1}^i = \frac{\beta\alpha}{1 - \beta\alpha} c_t^i \quad (33)$$

$$c_t^i = (1 - \beta\alpha)(1 - \tau) y_t^i \quad (34)$$

Similarly, aggregate capital investment ( $k_{t+1}$ ), consumption ( $c_t$ ) and output ( $y_t$ ) grow at the same rate, as can be easily seen by aggregating (33) and (34).

What make the current model different from the standard  $Ak$  growth models that we often see in the literature? First, the growth rate of the current period capital investment of an individual is lower than that of the

previous periods. The intuition is that in a growing economy, the level of current capital associated to an individual is higher than that of the previous periods. When this is coupled with diminishing returns to capital investment and capital markets imperfection, it is intuitive that the former grows slower than the latter. Second, two different family dynasties (say  $i$  and  $j$ ) follow different growth paths in our model. Individual households with a lower levels of initial capital experience a relatively higher growth rates of capital due to the imperfection in the capital markets and diminishing returns to investment. Finally, because of these two, there are two types of inequality in the current model: cross sectional inequality at a given moment of time and intergenerational inequality – the variation of income or wealth across generations.

The behavior of the  $Ak$  economy at individual could be quite different from that of the aggregate. This is because that individuals' production functions face a marginal diminishing returns to capital whereas the aggregate economy features a constant-return. Moreover, aggregate variables are explicitly involved in the evolution of individual households due to the externality effects of aggregate capital on households' productions. Therefore, the dynamics at individual level has some peculiar features that are not reflected at aggregate level.

Dividing (28) by (29), and using (27), we obtain

$$\begin{aligned}
c_{t+1}^i/k_{t+1}^i &= (c_t^i/k_t^i) \left( f'_{\varphi_{t+1}^i}(\varphi_{t+1}^i) / f'_{\varphi_t^i}(\varphi_t^i) \right) \\
&\Leftrightarrow \gamma_{t+1,c}^i = \gamma_{t+1,k}^i + \ln \left( f'_{\varphi_{t+1}^i}(\varphi_{t+1}^i) / f'_{\varphi_t^i}(\varphi_t^i) \right) \quad (35)
\end{aligned}$$

where  $\gamma_{t+1,k}^i \equiv \ln(k_{t+1}^i/k_t^i)$ .<sup>13</sup> And, from (27),

$$\gamma_y^i = (1 - \alpha) \gamma_k + \alpha \gamma_k^i \quad (36)$$

From (34) and (35) (or (36)), during the transition period, (i) an individual's consumption and income grow at the same but (ii) different rates from that of capital, depending on the relative position of the particular individual in the economy:

**Proposition 3.**  $\gamma_k^i > \gamma_c^i = \gamma_y^i$  during transition if the  $i$ th individual capital is below average ( $k_t^i < k_t$ ). Otherwise,  $\gamma_k^i < \gamma_c^i = \gamma_y^i$ . The inequality holds if the  $i$ th person represents the average person.

**Proof.** It is evident from (34) that  $\gamma_y^i = \gamma_c^i$ . Rewrite (36) as  $\gamma_y^i = \gamma_k^i + (1 - \alpha)(\gamma_k - \gamma_k^i)$ . Thus,  $\gamma_k^i \leq \gamma_y^i$  iff  $\gamma_k \geq \gamma_k^i$ . But, from (29),  $\gamma_k \geq \gamma_k^i \Leftrightarrow \ln k_{t+1}/k_t \geq \ln k_{t+1}^i/k_t^i$  iff  $k_t \geq k_t^i$ . ■

### 5.3. Aggregate dynamics

The dynamics of aggregate capital and inequality are shown in (31) and (32). We can also aggregate the Euler equation (28), first, by applying (A.6),

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<sup>13</sup>In what follows, we leave off the time subscript from the gamma variables when no confusion arises. For instance, we write  $\gamma_k^i$  instead of  $\gamma_{t+1,k}^i$ .

$$\ln(c_{t+1}/c_t) = 0.5 (\sigma_{t+1,c}^2 - \sigma_{t,c}^2) + \text{E} \left[ \ln \left( \beta \alpha (1 - \tau) (k_{t+1}^i/k_{t+1})^{\alpha-1} \right) \right] \quad (37)$$

Then, considering (A.3) and (32), we get,

$$\gamma_{t+1,c} \equiv \ln(c_{t+1}/c_t) = \beta \alpha (1 - \tau) \exp \{ 0.5 \alpha^3 (\alpha - 1) \sigma_{t,k}^2 \} \quad (38)$$

From (31) and (38), we obtain,

$$c_{t+1}/k_{t+1} = (c_t/k_t) \exp \{ ((\alpha + 1) \alpha^2 + 1 - \alpha) (\alpha - 1) 0.5 \sigma_{t,k}^2 \} \quad (39)$$

Therefore, the aggregate consumption-capital ratio ( $c_t/k_t$ ) is not constant, as  $\sigma_{t,k}^2$  continues to evolve during the transition period.

The growth rate of output is given by, from aggregating (27),

$$\gamma_{t+1,y} = \beta \alpha (1 - \tau) \exp \{ \alpha^3 (\alpha - 1) 0.5 \sigma_{t,k}^2 \} \quad (40)$$

considering that  $\text{E} [(k_t^i/k_t)^\alpha] = \exp \{ \alpha (\alpha - 1) 0.5 \sigma_{t,k}^2 \}$ .

**Proposition 4.** *During the transition period, when the capital markets are incomplete and  $\sigma_{t,k}^2 \neq 0$ , the aggregate economy of the Ak model features:  $\gamma_{t+1,c} = \gamma_{t+1,y} > \gamma_{t+1,k}$ . In the steady state, the economy remains in a balanced growth path where all variables grow at the same rate and inequality vanishes ( $\sigma_{t,k}^2 = 0$ ).*

**Proof.** Compare and contrast eqs. (31), (38) and (40) while noting (32). ■



## 6. Conclusion

The  $Ak$  model arguably is the basis of every endogenous growth models. But the model is often criticized for a lack of transitional dynamics. This paper has argued that the popular  $Ak$  models lack transitional dynamics merely due to the imposition of the convenient but less realistic conditions of a representative firm or household, and a perfect capital market, however.  $Ak$  models that follow Romer and Barro displays transitional dynamics under the conditions that the capital markets are imperfect and households are heterogeneous in terms of initial capital endowment. The class of  $Ak$  models are, especially, identified with diminishing-returns at individual but constant-returns at aggregate due to the presence of an externality effects.

A perfect capital market or a representative agent is associated to an instantaneous equalization of intra- and inter-temporal individual households' productivity in the economy. It fixes the rate of return to capital that leads the economy to converge to its long-run equilibrium path without transition. Capital markets imperfection in a heterogeneous environment that is coupled with diminishing returns to capital, on the other hand, makes initial individuals' productivity differences to persist across time and space. Consequently, there will be two types of inequalities in the economy: cross sectional inequality at a given moment of time and intergenerational inequality – the variation of income or wealth across generations. These have a direct impact on the economy, which, in turn, has influence on households' productivity through the externality effects. The process leads to transitional dynamics.

During the transition period of the economy, aggregate consumption and output grow at the same rate but higher than that of capital whereas individual consumption and output grow at the same rate but different from that of capital.

## A. Aggregation and distribution

In this appendix, we derive the aggregate Euler equation and the distributional dynamics, as shown in (24) and (26), under the conditions of incomplete markets and heterogeneous households.

### A.1. Distribution dynamics

In deriving the dynamics of inequality, first, rewrite the budget constraints (2) as,

$$c_t^i = a(1 - \tau_y) y_t^i \tag{A.1}$$

$$k_{t+1}^i = (1 - a)(1 - \tau_y) y_t^i \tag{A.2}$$

where  $a \in (0, 1)$ . Then, from (A.1), (A.2) and (8), it is straightforward to derive the following relation:

$$\sigma_{t,c}^2 = \sigma_{t+1,k}^2 = \text{var} [\ln f(\varphi_t^i)] \tag{A.3}$$

where  $\sigma_{t,c}^2 \equiv \text{var} [\ln c_t^i]$  and  $\sigma_{t+1,k}^2 \equiv \text{var} [\ln k_{t+1}^i]$ .

But, from a lognormal-normal relationship,

$$\text{var} [\ln f(\varphi_t^i)] = 2 \ln \text{E} [f(\varphi_t^i)] - 2 \text{E} [\ln f(\varphi_t^i)] \quad (\text{A.4})$$

then, substituting (A.4) into (A.3), we obtain,

$$\sigma_{t+1,k}^2 = 2 \ln \text{E} [f(\varphi_t^i)] - 2 \text{E} [\ln f(\varphi_t^i)] \quad (\text{A.5})$$

which characterizes the distributional dynamics of the model. We can rewrite (A.5), considering Lemma 1 and the subsequent arguments, as (24).

### A.2. Aggregation

Note that, first, since  $c_{t+1}^i/c_t^i$ , considering (A.1), has a lognormal distribution, we have

$$\ln(c_{t+1}/c_t) = \text{E} [\ln(c_{t+1}^i/c_t^i)] + 0.5(\sigma_{t+1,c}^2 - \sigma_{t,c}^2) \quad (\text{A.6})$$

where  $c_t \equiv \int_i c_t^i = \text{E}[c_t^i]$ . Then, aggregating the Euler equation is given by, using (6) and (8),

$$\text{E} [\ln c_{t+1}^i/c_t^i] = \delta^{-1} \ln(\beta(1-\tau)) + \delta^{-1} \text{E} [\ln f'_\varphi(\varphi_{t+1}^i)] \quad (\text{A.7})$$

Substituting (A.6) into (A.7), we obtain,

$$\ln(c_{t+1}/c_t) = 0.5 (\sigma_{t+1,c}^2 - \sigma_{t,c}^2) + \delta^{-1} \ln(\beta(1-\tau)) + \delta^{-1} \mathbb{E} [\ln f'_\varphi(\varphi_{t+1}^i)]$$

(A.8)

After using (A.3), we then have,

$$\ln(c_{t+1}/c_t) = 0.5 (\sigma_{t+2,k}^2 - \sigma_{t+1,k}^2) + \delta^{-1} \ln(\beta(1-\tau)) + \delta^{-1} \mathbb{E} [\ln f'_\varphi(\varphi_{t+1}^i)]$$

(A.9)

Finally, substituting (24) into (A.9) yields

$$\ln(c_{t+1}/c_t) = 0.5 (l(\sigma_{t+1,k}^2, \cdot) - \sigma_{t+1,k}^2) + \delta^{-1} \ln(\beta(1-\tau)) + \delta^{-1} s(\sigma_{t+1,k}^2, \cdot)$$

(A.10)

where  $s(\sigma_{t+1,k}^2, \cdot) = \mathbb{E} [\ln f'_\varphi(\varphi_{t+1}^i)]$ , considering Lemma 1 and the subsequent arguments.

## References

- Acemoglu, D., Ventura, J., 2002. The world income distribution. *The Quarterly Journal of Economics* 117 (2), 659–694.
- Aghion, P., Caroli, E., Garcia-Penalosa, C., 1999. Inequality and economic growth: The perspective of the new growth theories. *Journal of Economic Literature* 37 (4), 1615–1660.

- Aghion, P., Howitt, P., 1998. *Endogenous growth theory*. Cambridge and London: MIT Press.
- Angeletos, G.-M., Calvet, L.-E., 2006. Idiosyncratic production risk, growth and the business cycle. *Journal of Monetary Economics* 53 (6), 1095–1115.
- Angeletos, G.-M., Panousi, V., 2009. Revisiting the supply side effects of government spending. *Journal of Monetary Economics* 56 (2), 137–153.
- Arrow, K. J., 1962. The economic implications of learning by doing. *Review of Economic Studies* 29 (June), 155–173.
- Barro, R. J., 1990. Government spending in a simple model of endogeneous growth. *The Journal of Political Economy* 98 (5), S103–S125.
- Barro, R. J., Sala-i Martin, X., 2004. *Economic Growth*, 2nd Edition. The MIT Press, Cambridge, Massachusetts London, England.
- Benabou, R., 1996. Inequality and growth. National Bureau of Economic Research, Inc, NBER Working Papers: 5658.
- Benabou, R., 2000. Unequal societies: Income distribution and the social contract. *The American Economic Review* 90 (1), 96–129.
- Benabou, R., 2002. Tax and education policy in a heterogeneous-agent economy: What levels of redistribution maximize growth and efficiency? *Econometrica* 70 (2), 481–517.

- Boucekkine, R., Licandro, O., Puch, L. A., del Rio, F., 2005. Vintage capital and the dynamics of the ak model. *Journal of Economic Theory* 120 (1), 39–72.
- Carroll, C. D., Overland, J., Weil, D. N., 1997. Comparison utility in a growth model. *Journal of Economic Growth* 2 (4), 339–367.
- Carroll, C. D., Overland, J., Weil, D. N., 2000. Saving and growth with habit formation. *The American Economic Review* 90 (3), 341–355.
- Caselli, F., Ventura, J., 2000. A representative consumer theory of distribution. *The American Economic Review* 90 (4), 909–926.
- Frankel, M., 1962. The production function in allocation and growth: A synthesis. *The American Economic Review* 52 (5), 996–1022.
- Galor, O., Zeira, J., 1993. Income distribution and macroeconomics. *The Review of Economic Studies* 60 (1), 35–52.
- Gomez, M. A., 2008. Convergence speed in the ak endogenous growth model with habit formation. *Economics Letters* 100 (1), 16–21.
- Guerrini, L., 2010. Transitional dynamics in the ramsey model with ak technology and logistic population change. *Economics Letters* 109 (1), 17–19.

- Jones, L. E., Manuelli, R. E., 1990. A convex model of equilibrium growth: Theory and policy implications. *Journal of Political Economy* 98 (5), 1008–1038.
- Loury, G. C., 1981. Intergenerational transfers and the distribution of earnings. *Econometrica* 49 (4), 843–867.
- Mankiw, N. G., Romer, D., Weil, D. N., 1992. A contribution to the empirics of economic growth. *Quarterly Journal of Economics* 107 (2), 407–437.
- Neumann, J. V., 1937. "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes." *Ergebnisse eines Mathematischen Kolloquiums*, 8, translated by Karl Menger as "A model of general economic equilibrium". *The Review of Economic Studies* (1945) 13 (1), 1–9.
- Rebelo, S., 1991. Long-run policy analysis and long-run growth. *Journal of Political Economy* 99 (3), 500–521.
- Romer, P. M., 1986. Increasing returns and long-run growth. *Journal of Political Economy* 94 (5), 1002–1037.
- Tamura, R., 1991. Income convergence in an endogenous growth model. *Journal of Political Economy* 99 (3), 522–540.

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