*RUM : a Simple **R**ecursive **U**pdate **M**odel*

Providing a Condensed Representation

of a Putty-Semi-Putty Vintage Model

by

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1 Introduction

In this paper we present an extension of the Quasi-Putty-Putty model and the CRAPP model which were presented in van Zon (1992.D) and van Zon(1993), respectively. CRAPP stands for 'Continuous Recursive Adjustment Putty Putty' model. The latest version of the latter model is called 'RUM' which stands for 'Recursive Update Model', a name which is much more general than CRAPP, but which has a somewhat more 'serious' ring to it. RUM should be part of a larger framework incorporating two other models called 'MASTER' and 'SAM'.[1]

The principle reasons why we developed CRAPP and RUM are **first** that we feel that technological change does not fall as manna from heaven : it must be bought and paid for, **secondly** that many of the improvements in production processes are linked with embodied technological change, and **third** that there 'should' be a 'practical way' to incorporate the idea of the embodiment of technological change and the 'technology induced' scrapping of old equipment into a vintage framework, without having to engage in cumbersome calculations involving a complete 'bookkeeping' account of all individual vintages which have come into existence 'almost from the beginning of time'.[2]

The RUM model is based on a putty-semi-putty vintage structure, just like the CRAPP model. The principle difference between the CRAPP model and the RUM model is that RUM takes account of expectations regarding future prices and disembodied technological change in order to determine the best course of action needed to be taken today, both conditional on what has happened in the past and 'in response' to what is expected to happen in the future. Thus, present decisions build on decisions taken in the past, and are in line with the anticipations of future decisions. In short, we derive the principal

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1 MASTER is short for 'Model for the Analysis of Sectoral Technology and Employment Relations', while SAM stands for 'Skill Allocation Module'. For (some of) the main features of MASTER and SAM, see van Zon and Muysken (1992.A-C) and van Zon (1993).

2 Such an approach is 'standard practice' in the Dutch empirical vintage modelling tradition. See Den Hartog and Tjan (1976), Kuipers and van Zon (1982), Gelauff, Wennekers and de Jong (1985), and Muysken and van Zon (1987), for instance. A notable exception is Eigenraam (1987). The third motivation for the construction of CRAPP and RUM follows from the need to economise somewhat on vintage details, since the MASTER model, of which RUM should be an intrinsic part, focuses first and foremost on the heterogeneity of labour in the sectoral, educational and job-dimension. This three dimensional heterogeneity of labour already accounts for an enormous computational workload, to which SAM, the skill allocation module, being implemented as a series of large linear programming problems to be solved within the context of a Gauss-Seidel iterative procedure which is used to solve the MASTER model, adds a huge computational burden of its own.

features of the RUM model in an intertemporal setting, whereas the features of the CRAPP model were derived for the case of myopic behaviour. Nonetheless, both RUM and CRAPP perform in a strikingly similar way.

Why then is the transition from myopic behaviour to non-myopic behaviour important? The reason is that we assume that substitution possibilities between labour and capital before the moment of installation of a new piece of machinery are larger than the substitution possibilities after the moment of installation. This implies that the choice of an initial technique uniquely defines the entire ex-post iso-quant (as long as different iso-quants do not have any techniques in common). But since substitution possibilities ex-post are described by the ex-post unit-isoquant, this also implies that the ability to react to future (expected) changes in wage rates is also directly affected by the technical characteristics of the initial factor-mix. More in particular, the choice of a high labour/capital ratio in response to present wage conditions, for instance, diminishes ones opportunities to avoid future rises in wage costs which are associated with rises in wage-rates.

As the name already suggests, RUM uses a set of recursive update rules which describes the evolution over time of aggregate capital productivity and the aggregate capital/labour ratio in function of the ex-post substitution characteristics of the 'old' machinery and of the new machinery just installed. The advantage of using the RUM model rather than a full putty-semi-putty vintage model lies in the relative ease by which it can be handled : instead of tracing individual vintages for a considerable length of time, RUM makes positive use of the fact that it is often not necessary to know all the details of every individual vintage. From a macro-economic point of view, only the average characteristics of the vintage capital stocks are usually of any practical relevance. Hence, when we would be able to derive a small number of 'update rules' which describe changes in the average characteristics of the vintage capital stock both in terms of the changes in the technological characteristics of new equipment (due to factor substitution ex-ante and embodied technological change) and in terms of the changes in the characteristics of the 'old' equipment (due to factor substitution ex-post and due to disembodied technological change), then we would arrive at a condensed representation of all the relevant vintage detail of a putty-semi-putty model, without having to engage in cumbersome bookkeeping exercises. Both CRAPP and RUM have been designed to avoid these bookkeeping exercises. Nonetheless, both models are still able to imitate a full putty-semi-putty vintage model nearly perfectly.

The set-up of this paper is as follows. In section 2 we will describe the features of the RUM model, and section 3 will be used to sketch the outcomes of some simulation experiments using RUM as well as a full putty-semi-putty vintage ('bench-mark') model. Section 4 contains a summary and some concluding remarks.

2 The RUM Model

2.1 General Assumptions

We assume that there are two basic factors of production, labour and capital, which can be substituted both ex-ante and ex-post. We also assume that substitution possibilities are 'smooth' and that there are no costs involved in switching from the one technique to another. Nor are there any costs involved in switching from the one technology to another. A further assumption is that the diffusion of new technologies is an instantaneous process : every single producer invests in the newest technology only. At the same time we assume that capital costs are sunk costs ex-post.

Because of the smooth substitution possibilities ex-post, it follows that output can be produced using all the technologies which have come into existence from time immemorial. The reason is that it is possible to increase the marginal productivity of labour (and thus decrease variable costs per unit of output) indefinitly for any production function which obeys the Inada conditions. This will become more clear below.

We furthermore assume that entrepreneurs are price-takers on the factor markets as well as the output market. Moreover, we assume that producers form expectations using a partial adjustment scheme regarding expected rates of growth. These expectations are important considering the fact that, given the (more) limited substitution possibilities ex-post, entrepreneurs can only try to avoid the cost-consequences of a change in future wage-rates in as far as these latter changes have been anticipated. We will first discuss the myopic foresight case, however, and then turn to a description of the intertemporal setting. The technology is the same in both cases.

2.2 RUM and Myopic Foresight

For the ex-ante production function as well as the ex-post production function we use linear homogeneous CES functions. Denoting the level of output associated with vintage i at time t by $Y_{i,t}$, the amount of labour associated with vintage i at time t by $N_{i,t}$ and the amount of investment associated with vintage i at time t by $I_{i,t}$, we have :

$$Y_{t,t} = \left\{ A_t^a \cdot (N_{t,t})^{-\rho_a} + B_t^a \cdot (I_{t,t})^{-\rho_a} \right\}^{-1/\rho_a} \tag{1}$$

where the super-/subscript a denotes the ex-ante function. A^a_t and B^a_t are the CES distribution parameters and $\sigma_a = 1/(1+\rho_a)$ is the ex-ante elasticity of substitution. Similarly, for the ex-post production function we have :

$$Y_{i,t} = \left\{ A_{i,t}^p \cdot (N_{i,t})^{-\rho_p} + B_{i,t}^p \cdot (I_{i,t})^{-\rho_p} \right\}^{-1/\rho_p} \tag{2}$$

where t>i, and where the super- / subscript p denotes the ex-post parameters. Note that the main difference between (1) and (2) lies in the specification of the distribution parameters A and B. In the ex-ante case A depends on the time of installation only (embodied technical change stops at the moment of installation), whereas in the ex-post case A and B depend also on the time of observation: disembodied technical change takes over from embodied technical change (with a possibly different rate) from the moment of installation of a piece of equipment. Hence, with respect to the ex-ante distribution parameters we assume that :

$$A_t^a = A_0 \cdot (1+\mu_n)^{-\rho_a \cdot t} \tag{3}$$

$$B_t^a = B_0 \cdot (1+\mu_I)^{-\rho_a \cdot t}$$

where μ_n and μ_I are the rates of embodied labour and capital augmenting technological change, respectively. Because disembodied technological change has productivity effects which can be reaped only after the moment of installation, the rates of disembodied labour and capital augmenting technological change do not enter the decision framework regarding the choice of the optimum factor-mix in the myopic foresight case. Hence, given the assumptions of sunk capital costs ex-post and 'smooth' substitution possibilities both ex-ante and ex-post, it follows that the instantaneous cost minimisation problem which a producer faces can be written as :

$$\begin{aligned} C'_t = {} & w_t \cdot \sum_{j=-\infty}^{t} N_{j,t} + u_t \cdot I_{t,t} \\ & + \lambda^x \cdot \left(X_t - \sum_{j=-\infty}^{t} Y_{j,t} \right) \\ & + \sum_{j=-\infty}^{t-1} \lambda_j^y \cdot (Y_{j,t} - F^p(N_{j,t}, I_{j,t})) \\ & + \lambda_t^y \cdot (Y_{t,t} - F^a(N_{t,t}, I_{t,t})) \\ & + \sum_{j=-\infty}^{t} \lambda_j^I (I_{j,t} - I_{j,j} \cdot (1-\delta)^{t-j}) \end{aligned} \tag{4}$$

where w_t is the current wage rate, and u_t is the cost of using a unit of capital. Moreover $F^a()$ and $F^p()$ are short hand notations for the ex-ante production function and the ex-post production function, respectively. X_t is the total amount of output to be produced on both new equipment and old equipment. δ is the technical decay parameter (we assume depreciation by 'radioactive decay'). λ_j^y , λ_j^I and λ^x are the Lagrange multipliers associated with, the ex-post (j<t) and ex-ante (j=t) production functions, the decay equations with respect to investment, and the aggregate production capacity constraint, respectively.

Minimisation of (4), gives rise to the following first order conditions :

$$\frac{\partial C'_t}{\partial N_{j,t}} = w_t - \lambda_j^y \cdot \frac{\partial F^p}{\partial N_{j,t}} = 0 \tag{5}$$

$$\frac{\partial C'_t}{\partial N_{t,t}} = w_t - \lambda_t^y \cdot \frac{\partial F^a}{\partial N_{t,t}} = 0$$

$$\frac{\partial C'_t}{\partial I_{t,t}} = u_t - \lambda_t^y \cdot \frac{\partial F^a}{\partial I_{t,t}} = 0$$

$$\frac{\partial C'_t}{\partial Y_{j,t}} = \lambda_j^y - \lambda^x = 0$$

$$\frac{\partial C'_t}{\partial Y_{t,t}} = \lambda_t^y - \lambda^x = 0$$

where we have assumed the existing capital stock to be fixed.

Due to the linear homogeneity of $F^a()$, it immediately follows that λ_t^y is equal to unit total production cost of the new technology, i.e. :

$$\lambda_t^y = w_t \cdot \nu_{t,t} + u_t \cdot \kappa_{t,t} \tag{6}$$

$$\nu_{j,t} = \frac{N_{j,t}}{Y_{j,t}} \quad \forall j \leq t$$

$$\kappa_{j,t} = \frac{I_{j,t}}{Y_{j,t}} \quad \forall j \leq t$$

From equations (5) and (6) it follows that the optimum allocation of labour (and output) between 'new' equipment and 'old' equipment is determined by the condition that :

$$\lambda_j^y = \frac{w_t}{\frac{\partial Y_{j,t}}{\partial N_{j,t}}} = \frac{w_t \cdot \Delta N_{j,t}}{\frac{\partial Y_{j,t}}{\partial N_{j,t}} \cdot \Delta N_{j,t}} = w_t \cdot \nu_{t,t} + u_t \cdot \kappa_{t,t} \tag{7}$$

(7) says that the marginal unit of output produced on old equipment should be produced at a variable cost which is equal to the unit total cost of output on new equipment. This is exactly what the Malcomson scrapping condition says with regard to the optimum vintage composition of the capital stock in a situation of cost-minimisation (c.f. Malcomson (1975) and van Zon and Muysken (1992B)). It follows furthermore from (5) that for a minimum of (4) to be reached, the marginal labour productivity of every single existing vintage should be equal to the marginal productivity of labour on the new vintage. Moreover, for the new vintage we should have:

$$\frac{\left(\frac{\partial Y_{t,t}}{\partial N_{t,t}}\right)}{\left(\frac{\partial Y_{t,t}}{\partial I_{t,t}}\right)} = \frac{w_t}{u_t} \tag{8}$$

Using the ex-ante function (1), we have :

$$\frac{\partial Y_{t,t}}{\partial N_{t,t}} = A_t^a \cdot \left\{\frac{Y_{t,t}}{N_{t,t}}\right\}^{\frac{1}{\sigma_a}} \tag{9}$$

$$\frac{\partial Y_{t,t}}{\partial I_{t,t}} = B_t^a \cdot \left\{\frac{Y_{t,t}}{I_{t,t}}\right\}^{\frac{1}{\sigma_a}}$$

which, combined with (5) and (6), leads to:

$$\nu_{t,t} = \kappa_{t,t} \cdot \left\{\frac{B_t^a \cdot w_t}{A_t^a \cdot u_t}\right\}^{-\sigma_a} = \kappa_{t,t} \cdot h_t \tag{10}$$

where h_t is implicitly defined by (10). Using (1), (6) and (10), we find :

$$\kappa_{t,t} = \left\{A_t^a \cdot h_t^{-\rho_a} + B_t^a\right\}^{\frac{1}{\rho_a}} \tag{11}$$

$$\nu_{t,t} = \left\{A_t^a + B_t^a \cdot h_t^{\rho_a}\right\}^{\frac{1}{\rho_a}}$$

Hence, from (10) it follows that the optimum labour intensity is determined by the wage-rental ratio, as well as the distribution parameters of the CES ex-ante function next to the ex-ante elasticity of substitution. This also goes for the technical coefficients (c.f. (11)).

In figure 1 below, the ex-ante unit iso-quant has been labelled e.a., while two of the ex-post iso-quants have been labelled e.p. The ex-ante iso-quant has been drawn as an envelope of all possible ex-post iso-quants.[3] We have also drawn two different wage rental ratios, which give rise to two different optimum values of the labour intensity of production on new (and old) equipment. Suppose that at time 0 the ruling wage rental ratio is such that point A would be chosen. Then, at time 1, the wage rental ratio changes such that on new equipment point B becomes optimum. With the rise in the relative wage rate, the labour / capital ratio has a tendency to fall. But on old equipment, substitution possibilities between labour and capital are more limited by assumption, and therefore the rise in the relative wage rate invokes only a moderate adjustment of the labour / capital ratio on old equipment. This is depicted by the move from point A to point C along the ex-post iso-quant, as opposed to the 'move' from point A to point B along the ex-ante iso-quant, where point B is the optimum capital labour combination for equipment installed at time 0 at the new price vector. Obviously, when substitution possibilities ex-post would be equal to those ex-ante, the distinction between old equipment and new equipment vanishes entirely in the absence of embodied technological change.

3 Note that for the type of embodied technical change we have assumed, the entire iso-quant field would shift towards the origin. This 'movement' is caused by a 'technological pull' of both axes. The rates of factor augmenting technological change indicate the relative strengths of the technology 'forces' acting upon the iso-quant field.

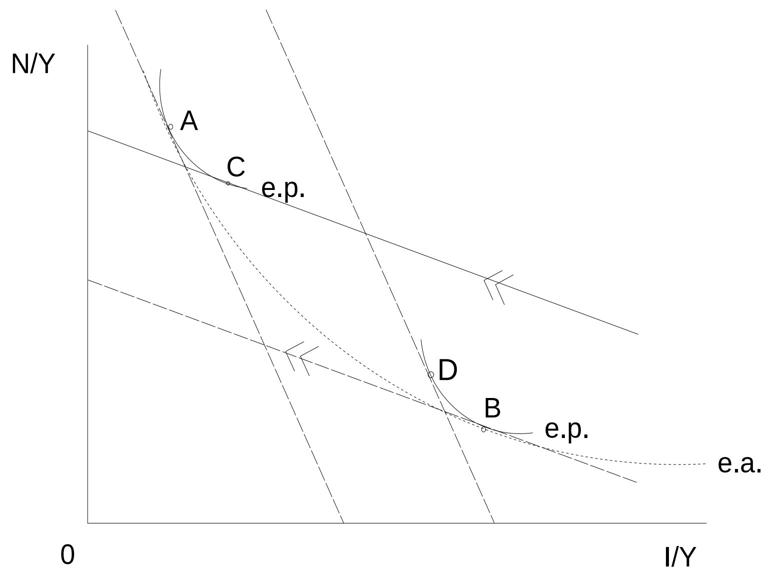

Fig. 1 The ex-ante production function envelope and myopic foresight

Note that by choosing a specific technique on one of the infinitely many ex-post iso-quants which are associated with a certain ex-ante technology, one also chooses one's future substitution possibilities with respect to the vintage under consideration. Hence, when one would expect the 'average future relative wage rate' to be given by the slope of the straight line through B, while the initial relative wage rate is given by slope of the straight line through A, one would probably do better choosing the ex-post iso-quant implicitly defined by point B rather than the one which is implicitly defined by point A. We will come back to this in more detail below.

2.3 Linking Ex-Ante and Ex-Post Substitution Characteristics

From figure 1 above it is clear that the exact location of an ex-post iso-quant in the N / Y, I / Y plane depends, among other things, on the value of the wage / rental ratio. More specifically, the notion of the ex-ante iso-quant as an envelope of ex-post unit iso-quants can help to derive the values of the ex-post CES distribution parameters A^p_i and B^p_i in function of the substitution parameters ex-ante and ex-post as well as the ex-ante CES distribution parameters. For, the figure shows that for the ex-post production function two conditions should hold in order for the ex-post iso-quant in question to be consistent with the ex-ante choice set :

1 the tangential technique (denoted by $(\bar{\nu}, \bar{\kappa})$) should be part of the ex-ante envelope as well as part of its associated ex-post unit iso-quant ;

2 since the envelope has only one technique in common with each ex-post unit iso-quant, and since substitution possibilities ex-ante and ex-post are 'smooth' by assumption, it follows that for the tangential technique $(\bar{\nu},\bar{\kappa})$ the slopes of both the ex-ante iso-quant and the ex-post iso-quant should be the same.

From requirement 2 it follows for the tangential technique $(\bar{\nu},\bar{\kappa})$ that :

$$\left(\frac{d\bar{\nu}}{d\bar{\kappa}}\right)^{ex\ ante}=\left(\frac{d\bar{\nu}}{d\bar{\kappa}}\right)^{ex\ post}\quad\Rightarrow\quad\frac{B_t^a}{A_t^a}\cdot\left\{\frac{\bar{\kappa}}{\bar{\nu}}\right\}^{-1/\sigma_a}=\frac{B_{t,t}^p}{A_{t,t}^p}\cdot\left\{\frac{\bar{\kappa}}{\bar{\nu}}\right\}^{-1/\sigma_p}\tag{12}$$

which leads directly to :

$$A_{t,t}^p=\frac{B_{t,t}^p}{B_t^a}\cdot A_t^a\cdot\left(\frac{\bar{\nu}}{\bar{\kappa}}\right)^{\rho_p-\rho_a}\tag{13}$$

From requirement 1 it follows moreover that the labour coefficient ex-post as well as the capital coefficient ex-post should be equal to their ex-ante counterparts. Hence, for the tangential technique $(\bar{\nu},\bar{\kappa})$ we should have :

$$A_{t,t}^p\cdot\bar{\nu}^{-\rho_p}+B_{t,t}^p\cdot\bar{\kappa}^{-\rho_p}=A_t^a\cdot\bar{\nu}^{-\rho_a}+B_t^a\cdot\bar{\kappa}^{-\rho_a}=1\tag{14}$$

Substitution of (13) into (14) yields :

$$B_{t,t}^p=B_t^a\cdot\bar{\kappa}^{(\rho_p-\rho_a)}\tag{15}$$

Furthermore, substitution of (15) into (13) yields in turn :

$$A_{t,t}^p=A_t^a\cdot\bar{\nu}^{(\rho_p-\rho_a)}\tag{16}$$

Equations (15) and (16) show that the ex-post distribution parameters are uniquely determined by the tangential technique $(\bar{\nu},\bar{\kappa})$. [4] Hence, by choosing a tangential

4 Note that embodied technological change has an influence on the ex-post CES distribution parameters through their dependence on the ex-ante parameters. Note also that, in the case of identical elasticities of substitution ex-ante and ex-post, the ex-post CES distribution parameters are identical to their ex-ante counterparts.

technique $(\bar{\nu}, \bar{\kappa})$, one also chooses a unique ex-post unit iso-quant in the process. We will now introduce equations (15) and (16) in a fully intertemporal optimization setting where one not only has to choose the optimum initial factor-coefficients ex-ante and ex-post, but also optimum (initial) ex-post unit iso-quants, by selecting the appropriate tangential techniques $(\bar{\nu}, \bar{\kappa})$.

2.4 RUM and Non-Myopic Foresight

2.4.1 Introduction

We now assume that entrepreneurs try to maximise the present value of all their productive activities now and in the future by :

1 allocating labour in the 'right' way among existing vintages and new vintages;

2 allocating labour and capital in the 'right' way to new vintages;

3 allocating investment in the 'right' way among new vintages.

The way in which entrepreneurs can try to achieve this goal, is **first** by selecting the capital-output and labour-output ratios on both existing and new vintages in accordance with their expectations regarding relative prices, **secondly** by selecting 'optimum' ex-post production technologies for new equipment, and **third** by selecting the volume of investment in new equipment. These choices are all conditional on an aggregate capacity output constraint (which may be based on expectations regarding demand, which are however not specified any further here). They are also conditional on the functional forms of the ex-post production functions and of the ex-ante production function, the latter of which defines the infinitely large family of ex-post iso-quants entrepreneurs have to pick their choice from.

2.4.2 On Choosing Optimum Factor Proportions

As usual, we assume input-prices to be given to individual entrepreneurs, while output prices are assumed exogenously given. [5] The expected present value of the firm is then given by :

5 The analysis does however not depend on the latter assumption. At this stage, we make this assumption for ease of exposition only.

$$\Phi_t = \sum_{j=-\infty}^{t-1} \sum_{i=t}^{\infty} p'_i \cdot \frac{I_{j,j} \cdot (1-\delta)^{i-j}}{\kappa_{j,i}} \cdot \left(1 - \frac{w_i}{p'_i} \cdot \nu_{j,i}\right) \tag{17}$$

$$+ \sum_{j=t}^{\infty} \sum_{i=j}^{\infty} p'_i \cdot \frac{I_{j,j} \cdot (1-\delta)^{i-j}}{\kappa_{j,i}} \cdot \left(1 - \frac{w_i}{p'_i} \cdot \nu_{j,i}\right)$$

$$- \sum_{j=t}^{\infty} q'_j \cdot I_{j,j}$$

$$+ \sum_{j=-\infty}^{t-1} \sum_{i=t}^{\infty} \lambda^o_{j,i} \cdot (f^{j,i}(\nu_{j,i}, \kappa_{j,i}) - 1)$$

$$+ \sum_{j=t}^{\infty} \sum_{i=j}^{\infty} \lambda^n_{j,i} \cdot (f^{j,i}(\nu_{j,i}, \kappa_{j,i}, \bar{\nu}_j, \bar{\kappa}_j) - 1)$$

$$+ \sum_{i=t}^{\infty} \lambda^x_i \cdot \left(X_i - \sum_{j=-\infty}^{i} \frac{I_{j,j} \cdot (1-\delta)^{i-j}}{\kappa_{j,i}}\right)$$

$$+ \sum_{j=t}^{\infty} \lambda^{gj} \cdot (1 - g_j(\bar{\nu}_j, \bar{\kappa}_j))$$

where p’, w’ and q’ are the expected present value of the price of a unit of output, of a unit of labour and of a unit of investment, respectively, and where $z'_i = z_i \cdot (1 + r_t)^{-(i-t)}$ for z=p,w,q. $f^{j,i}$ () is the linear homogeneous ex-post production function associated with vintage j at time i, while g^j() is the linear homogeneous ex-ante production function associated with vintage j at the time of its installation. Both $f^{j,i}$ and g^j have been defined in terms of ’technical coefficients’ rather than in terms of the absolute factor-inputs. The time index t represents the present (decision) moment, while the index j is associated with the time of installation of a specific vintage. Furthermore, i is a time index which refers to the present and the future.

The first term of Φ_t represents the expected present value of all the quasi-rents associated with the operation of the old vintages which were installed up to and including time t-1. The only variables under the control of producers in this case are the labour / output and capital / output ratios, since the amount of investment and the nature of investment (in terms of the (endogenous) distribution parameters of the associated ex-post production function) of these vintages have already been determined in the past. Hence :

$$\frac{\partial \Phi_t}{\partial \nu_{j,i}} = -w'_i \cdot Y_{j,i} + \lambda^o_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \nu_{j,i}} = 0 \qquad \forall j < t, i \geq t \tag{18.A}$$

$$\frac{\partial \Phi_t}{\partial \nu_{j,i}} = -w'_i \cdot Y_{j,i} + \lambda^n_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \nu_{j,i}} = 0 \qquad \forall j \geq t, i \geq j \tag{18.B}$$

$$\frac{\partial \Phi_t}{\partial \kappa_{j,i}} = -p'_i \cdot \frac{Y_{j,i}}{\kappa_{j,i}} \cdot \left(1 - \frac{w_i}{p_i} \cdot \nu_{j,i}\right) + \lambda^o_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} + \lambda^x_i \cdot \frac{Y_{j,i}}{\kappa_{j,i}} = 0 \qquad \forall j < t, i \geq t \tag{18.C}$$

$$\frac{\partial \Phi_t}{\partial \kappa_{j,i}} = -p'_i \cdot \frac{Y_{j,i}}{\kappa_{j,i}} \cdot \left(1 - \frac{w_i}{p_i} \cdot \nu_{j,i}\right) + \lambda^n_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} + \lambda^x_i \cdot \frac{Y_{j,i}}{\kappa_{j,i}} = 0 \qquad \forall j \geq t, i \geq j \tag{18.D}$$

$$\frac{\partial \Phi_t}{\partial I_{j,j}} = \sum_{i=j}^{\infty} p'_i \cdot \frac{(1-\delta)^{i-j}}{\kappa_{j,i}} \cdot \left(1 - \frac{w_i}{p_i} \cdot \nu_{j,i}\right) - q'_j - \sum_{i=j}^{\infty} \frac{(1-\delta)^{i-j}}{\kappa_{j,i}} \cdot \lambda^x_i = 0 \qquad \forall j \geq t, i \geq j \tag{18.E}$$

$$\frac{\partial \Phi_t}{\partial \bar{\nu}_j} = \sum_{i=j}^{\infty} \frac{\partial f^{j,i}}{\partial \bar{\nu}_j} \cdot \lambda^n_{j,i} - \lambda^g_j \cdot \frac{\partial g^j}{\partial \bar{\nu}_j} = 0 \qquad \forall j \geq t, i \geq j \tag{18.F}$$

$$\frac{\partial \Phi_t}{\partial \bar{\kappa}_j} = \sum_{i=j}^{\infty} \frac{\partial f^{j,i}}{\partial \bar{\kappa}_j} \cdot \lambda^n_{j,i} - \lambda^g_j \cdot \frac{\partial g^j}{\partial \bar{\kappa}_j} = 0 \qquad \forall j \geq t, i \geq j \tag{18.G}$$

where $Y_{j,i}$ is defined as the output associated with vintage j at time i, i.e. :

$$Y_{j,i} = \frac{I_{j,j} \cdot (1-\delta)^{i-j}}{\kappa_{j,i}} \qquad \forall i \geq j \tag{19}$$

Because of the linear homogeneity of $f^{j,i}$ in $\nu_{j,i}$ and $\kappa_{j,i}$, it follows from the application of the Euler-equation to equations (18.A) and (18.C) that :

$$\lambda^o_{j,i} = (p'_i - \lambda^x_i) \cdot Y_{j,i} \qquad \forall\, j < t, i \geq t \tag{20}$$

Likewise, from (18.B) and (18.D) it follows that :

$$\lambda^n_{j,i} = (p'_i - \lambda^x_i) \cdot Y_{j,i} \qquad \forall\, j \geq t, i \geq j \tag{21}$$

Substitution of equations (20) and (21) into (18.A) and (18.B), leads to the conclusion that all marginal labour productivities should be equal both for existing vintages and for vintages still to be installed, since the ratio $w'_i/(p'_i - \lambda^x_i)$ is independent of the time of installation of a particular vintage. From equation (18.D) we have :

$$p'_i \cdot \frac{(1-\delta)^{i-j}}{\kappa_{j,i}} \cdot \left(1 - \frac{w_i}{p_i} \cdot \nu_{j,i}\right) = \left(\lambda^n_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} + \lambda^x_i \cdot \frac{Y_{j,i}}{\kappa_{j,i}}\right) \cdot \frac{(1-\delta)^{i-j}}{Y_{j,i}} \tag{22}$$

Substitution of (22) into (18.E) leads directly to :

$$q'_j = \sum_{i=j}^{\infty} \lambda^n_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot \frac{(1-\delta)^{i-j}}{Y_{j,i}} = \sum_{i=j}^{\infty} (p'_i - \lambda^x_i) \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot (1-\delta)^{i-j} \tag{23}$$

by means of straightforward substitution of (21) into the first part of (23). Equation (23) can be used to obtain the value of $p'_j - \lambda^x_j$:[6]

$$p'_j - \lambda^x_j = q'_j \cdot \kappa_{j,j} \cdot \left(1 - \frac{(1-\delta)\cdot(1+\hat{q}_j)}{1+r_t}\right) + w'_j \cdot \nu_{j,j} \tag{24}$$

which, when substituted into (18.B) and (18.D) for i=j, gives :

$$\left(\frac{\partial f^{j,j}}{\partial \nu_{j,j}}\right) \Big/ \left(\frac{\partial f^{j,j}}{\partial \kappa_{j,j}}\right) = \frac{w'_j}{q'_j \cdot \left(1 - \frac{(1-\delta)\cdot(1+\hat{q}_j)}{1+r_t}\right)} = \frac{w'_j}{q''_j} \tag{25}$$

where q''_j is implicitly defined by (25) and where we have substituted equation (24), (18.B) and (18.D), and where $\hat{q}$ represents the (expected) rate of growth of the price-index of investment goods.[7] From (25) it follows immediately that :

$$\frac{A^p_{j,j} \cdot \nu_{j,j}^{-(1+\rho_p)}}{B^p_{j,j} \cdot \kappa_{j,j}^{-(1+\rho_p)}} = \frac{w'_j}{q''_j} \quad \Rightarrow \quad \nu_{j,j} = \kappa_{j,j} \cdot \left(\frac{B^p_{j,j} \cdot w'_j}{A^p_{j,j} \cdot q''_j}\right)^{-1/(1+\rho_p)} = h_j \cdot \kappa_{j,j} \tag{26}$$

where h_j is implicitly defined by (26). Substitution of (26) into the ex-post production function $f^{j,j}$ yields therefore :

$$\kappa_{j,j} = \left\{A^p_{j,j} \cdot h_j^{-\rho_p} + B^p_{j,j}\right\}^{1/\rho_p} \tag{27}$$

$$\nu_{j,j} = \left\{A^p_{j,j} + B^p_{j,j} \cdot h_j^{\rho_p}\right\}^{1/\rho_p}$$

where $A^p_{j,j}$ and $B^p_{j,j}$ depend on the tangential technique to be chosen from the ex ante function. Hence, the optimum values of the labour coefficients of both the new techniques and the existing techniques can not be determined yet.

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6 See Appendix A.

7 Note that q''_t is approximately equal to the user cost of capital as it is usually defined, i.e. $q''_t = q_t.(r_t + \delta - \hat{q})$.

2.4.3 Choosing the Optimum Ex-Post Iso-Quant

In a putty-clay vintage model of production, it is possible to ’condense’ information about the future into a present-value price system which is used to select the optimum ex-post iso-quant and the optimum entry point of the new technique on that iso-quant at the same time (see Meijers and van Zon (1991) for an explicit account of such a procedure in a multi-level CES dual cost-function setting, as well as Kuipers and van Zon (1982) and Muysken and van Zon (1987) for less explicit applications of the present value price system). The reason is simply that, in the case of a Leontief ex-post production function, the entry point must be on the ex-ante iso-quant and it will stay there indefinitly except for the influence of disembodied technical change. In a putty-semi-putty situation, however, labour coefficients ex-post can vary in response to changes in relative prices too. Moreover, given the optimality condition that at any point in time all marginal labour productivities should be equal, the future time-path of wage costs associated with a specific technique chosen today, is also influenced by what future technologies will look like in terms of their technical characteristics. [8] Hence, in this case, the impact of future circumstances on current decisions can not be condensed that easily into a ’pure’ present value price-system. Rather, the future needs to be integrated in the decision framework in a somewhat different way.

Equations (18.F) and (18.G) describe the marginal conditions which the identifying tangential techniques have to obey in order to ensure that profits are maximised, also in an intertemporal setting. Recall that :

$$f^{j,i} = \left\{ A^p_{j,i} \cdot \nu_{j,i}^{-\rho_p} + B^p_{j,i} \cdot \kappa_{j,i}^{-\rho_p} \right\}^{-1/\rho_p} = 1 \tag{28}$$

$$g^j = \left\{ A^a_j \cdot \bar{\nu}_j^{-\rho_a} + B^a_j \cdot \bar{\kappa}_j^{-\rho_a} \right\}^{-1/\rho_a} = 1$$

$$A^p_{j,i} = A^a_j \cdot \bar{\nu}_j^{-(\rho_p - \rho_a)} \cdot (1+\gamma_n)^{-\rho_p \cdot (i-j)}$$

$$B^p_{j,i} = B^a_j \cdot \bar{\kappa}_j^{-(\rho_p - \rho_a)} \cdot (1+\gamma_I)^{-\rho_p \cdot (i-j)}$$

$$A^a_j = A_0 \cdot (1+\mu_n)^{-\rho_a \cdot j}$$

$$B^a_j = B_0 \cdot (1+\mu_I)^{-\rho_a \cdot j}$$

8 Note that such a dependence on future technological characteristics of technological choices to be made today is also implied by the Malcomson scrapping condition in a full putty-clay vintage setting.

where γ_n and γ_I are the rates of disembodied labour and capital augmenting technical change, respectively. Using equations (28), (18.F) and (18.G), we have :

$$\frac{\partial f^{j,i}}{\partial \bar{\nu}_j} = -\left(\frac{1}{\rho_p}\right) \cdot \frac{\partial A^p_{j,i}}{\partial \bar{\nu}_j} = -\left(\frac{\rho_p - \rho_a}{\rho_p}\right) \cdot \frac{A^p_{j,i}}{\bar{\nu}_j} \cdot \nu_{j,i}^{-\rho_p} = -\left(\frac{\rho_p - \rho_a}{\rho_p}\right) \cdot \frac{\partial f^{j,i}}{\partial \nu_{j,i}} \cdot \frac{\nu_{j,i}}{\bar{\nu}_j} \tag{29}$$

$$\frac{\partial f^{j,i}}{\partial \bar{\kappa}_j} = -\left(\frac{1}{\rho_p}\right) \cdot \frac{\partial B^p_{j,i}}{\partial \bar{\kappa}_j} = -\left(\frac{\rho_p - \rho_a}{\rho_p}\right) \cdot \frac{B^p_{j,i}}{\bar{\kappa}_j} \cdot \nu_{j,i}^{-\rho_p} = -\left(\frac{\rho_p - \rho_a}{\rho_p}\right) \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot \frac{\kappa_{j,i}}{\bar{\kappa}_j}$$

Using (29) and the linear homogeneity of $f^{j,i}$ and g^j, and applying the Euler equation to (18.F) and (18.G), we obtain :

$$\begin{aligned}\lambda^g_j &= \lambda^g_j \cdot \left(\frac{\partial g^j}{\partial \bar{\nu}_j} \cdot \bar{\nu}_j + \frac{\partial g^j}{\partial \bar{\kappa}_j} \cdot \bar{\kappa}_j\right) = \sum_{i=j}^{\infty} \lambda^n_{j,i} \cdot \left(\frac{\partial f^{j,i}}{\partial \bar{\nu}_j} \cdot \bar{\nu}_j + \frac{\partial f^{j,i}}{\partial \bar{\kappa}_j} \cdot \bar{\kappa}_j\right) \\ &= -\left(\frac{\rho_p - \rho_a}{\rho_p}\right) \cdot \sum_{i=j}^{\infty} \lambda^n_{j,i} \cdot \left(\frac{\partial f^{j,i}}{\partial \nu_{j,i}} \cdot \nu_{j,i} + \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot \kappa_{j,i}\right) = -\left(\frac{\rho_p - \rho_a}{\rho_p}\right) \cdot \sum_{i=j}^{\infty} \lambda^n_{j,i}\end{aligned} \tag{30}$$

Using (29), (30), (18.B) and (18.E) it follows immediately that :

$$\frac{\partial g^j}{\partial \bar{\nu}_j} \cdot \bar{\nu}_j = \frac{\sum_{i=j}^{\infty} \frac{\partial f^{j,i}}{\partial \nu_{j,i}} \cdot \nu_{j,i} \cdot \lambda^n_{j,i}}{\sum_{i=j}^{\infty} \lambda^n_{j,i}} = \frac{\sum_{i=j}^{\infty} w'_i \cdot \nu_{j,i} \cdot Y_{j,i}}{\sum_{i=j}^{\infty} (p'_i - \lambda^x_i) \cdot Y_{j,i}} = \frac{\sum_{i=j}^{\infty} w'_i \cdot \nu_{j,i} \cdot (1-\delta)^{i-j}}{\sum_{i=j}^{\infty} (p'_i - \lambda^x_i) \cdot (1-\delta)^{i-j}} \tag{31}$$

which is equal to the output weighted average of the present value of unit labour costs over unit total costs on the newest vintage. The rightmost part of (31) rests on the assumption that the rate of decrease of $Y_{j,i}$ can be (roughly) approximated by the value of the decay parameter. [9] Note, however, that since the relative size of the error is the same in both the nominator and the denominator of (31), the errors cancel each other to some extent.

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9 Note that this is indeed a rough approximation, since the latter also assumes that the amount of labour allocated to the machinery in question would also have to fall at a rate equal to δ, since otherwise output could not fall at that rate (the ex-post production function is linear homogeneous by assumption). However, when wage costs on an old vintage rise more rapidly than average total costs on a new vintage, then marginal labour productivity on the old vintage should rise in compensation, and hence labour input should fall more rapidly than capital input. The ensuing rate of decrease of output would be somewhere in between the different rates of decrease of both inputs.

From (18.A), (20) and (29) we immediately obtain :

$$\nu_{j,i} = \left(\frac{w'_i}{A^p_{j,i} \cdot (p'_i - \lambda^x_i)} \right)^{-\sigma_p} = (A^p_{j,i})^{\sigma_p} \cdot \left(\frac{w'_i}{\psi_i} \right)^{-\sigma_p} \tag{32}$$

where ψ_i is implicitly defined by (32) and equal to unit total costs on the newest vintage. Substitution of (32) into (31) yields therefore :

$$\frac{\partial g^t}{\partial \bar{\nu}_t} \cdot \bar{\nu}_t = \frac{\sum_{i=t}^{\infty} (A^p_{t,i})^{\sigma_p} \cdot (w'_i)^{1-\sigma_p} \cdot \psi_i^{\sigma_p} \cdot (1-\delta)^{i-t}}{\sum_{i=t}^{\infty} \psi_i \cdot (1-\delta)^{i-t}} \tag{33}$$

Assuming constant rates of growth of the variables in (33), where applicable, we immediately obtain :

$$\frac{\partial g^t}{\partial \bar{\nu}_t} \cdot \bar{\nu}_t = \frac{(A^p_{t,t})^{\sigma_p} \cdot \left(\frac{w'_t}{\psi_t} \right)^{1-\sigma_p} \cdot \sum_{i=t}^{\infty} \left\{ (1+\hat{A}^p_{t,t})^{\sigma_p} \cdot (1+\hat{w}'_t)^{1-\sigma_p} \cdot (1+\hat{\psi}_t)^{\sigma_p} \cdot (1-\delta) \right\}^{i-t}}{\sum_{i=t}^{\infty} \{(1+\hat{\psi}_t) \cdot (1-\delta)\}^{i-t}} \tag{34}$$

Because $A^p_{t,i}$ depends explicitly on the 'tangential technique' , it follows that (34) can be rewritten as :[10]

$$A^a_{t,t} \cdot \bar{\nu}_t^{-\rho_a} = (A^p_{t,t})^{\sigma_p} \cdot Z_t = A^{a\ \sigma_p}_{t,t} \cdot \bar{\nu}_t^{-(\rho_p - \rho_a) \cdot \sigma_p} \cdot Z_t \tag{35}$$

where Z_t is a collection of terms implicitly defined by (34) and (35) taken together. Hence, from (35) we immediately obtain :

$$\bar{\nu}_t = \left\{ A^a_{t,t} \cdot Z_t^{-\frac{1}{1-\sigma_p}} \right\}^{\sigma_a} \tag{36}$$

As long as the growth of nominal wages and the growth of the price of investment is at most equal to the rate of interest, both the summations present in equation (34) have

10 Of course, ψ_t also depends on the tangential technique, but at this stage we ignore this, since the latter dependency is a more implicit one, and must be taken account of during the simultaneous solution of the model itself.

a finite value, because the multiplicative term in the geometric expansion is less than one. Hence, assuming that these terms are indeed less than one (and both the influence of technical progress and of technical decay support this assumption), it follows immediately that :

$$Z_t = \left(\frac{w'_t}{\psi_t}\right)^{1-\sigma_p} \cdot \frac{1-(1+\hat{\psi}_t)\cdot(1-\delta)}{1-(1+\gamma_n)^{-\rho_p\cdot\sigma_p}\cdot(1+\hat{w}_t)^{1-\sigma_p}\cdot(1+r_t)^{\sigma_p-1}\cdot(1+\hat{\psi})^{\sigma_p}\cdot(1-\delta)} \tag{37}$$

Hence, given the assumptions about the rates of change of factor prices relative to the interest rate, it also follows that the rate of change of discounted future unit costs will be negative, and therefore (37) will almost certainly hold true.

However, a minor problem still remains to be solved. The rate of growth of future unit production costs is not known. Hence, we propose to obtain an estimate of this rate of growth as follows. First assume $\hat{\psi}$ to be equal to zero, and calculate Z_t in accordance with (37). Then, given Z_t, it follows that the tangential technique $(\bar{\nu}_t, \bar{\kappa}_t)$ can be determined for all t, after which the entry-technique on the 'newest' ex-post production function is immediately obtained from (27). The unit cost on the newest vintage can then be determined, and hence, the optimum allocation of labour to existing vintages can be derived using (32). Given the value of ψ_{t-1}, it is possible to obtain the rate of growth of unit production cost directly. Now redo the previous calculations with this first round estimate of ψ_t and $\hat{\psi}_t$, and so on until convergence has been achieved. Then, given the ex-post production functions and the existing capital stock, it is possible to obtain total capacity output associated with the existing capital stock, after which the capacity gap to be filled by output from the newest equipment can be obtained. Given the optimum capital coefficient for new equipment, the amount of investment follows directly from the size of the capacity gap, and so does the required amount of labour associated with the newest vintage. Thus, we are able to arrive at aggregate capacity labour demand and aggregate 'capital' demand by summing over all vintages which are in existence at some moment of time. **The problem is that there are infinitely many vintages.** Hence, adding them all together in order to obtain aggregate capacity output and aggregate capacity labour demand is simply not possible. We therefore present a **practical shortcut** in the next section.

2.5 The Recursive Update Rules

From the first order conditions for a profit maximum (c.f. (18.A), (18.B), (20) and (21)) it follows that all marginal labour productivities should be the same for existing machinery and equipment and for new machinery. Using (18) we therefore have :

$$A^p_{t,t}\cdot\{\nu_{t,t}\}^{-1/\sigma_p} = A^p_{i,t}\cdot\{\nu_{i,t}\}^{-1/\sigma_p} \quad\Rightarrow\quad \nu_{i,t} = \nu_{t,t}\cdot(A^p_{t,t})^{-\sigma_p}\cdot(A^p_{i,t})^{\sigma_p} = \xi_t\cdot(A^p_{i,t})^{\sigma_p} \tag{38}$$

where ξ_t is implicitly defined by (38). (38) shows that the optimum value of the labour coefficient on an existing vintage consists of a vintage specific part and a general part. [11] The corresponding value of the capital coefficient can be obtained from the ex-post production function (c.f. (28)) :

$$\{\kappa_{i,t}\}^{-\rho_p} = \frac{1}{B^p_{i,t}} - \frac{A^p_{i,t} \cdot \{\nu_{i,t}\}^{-\rho_p}}{B^p_{i,t}} \tag{39}$$

Using (38) and (39), we immediately obtain :

$$\zeta_{i,t} = \left\{\frac{1}{\kappa_{i,t}}\right\}^{\rho_p} = \frac{1}{B^p_{i,t}} - (\xi_t)^{-\rho_p} \cdot \frac{(A^p_{i,t})^{\sigma_p}}{B^p_{i,t}} \tag{40}$$

Equation (40) provides one of the central equations of the RUM model. Note that $\zeta_{i,t}$ is implicitly defined as the capital productivity of vintage i at time t, raised to the power of ρ_p. Let us now define :

$$\bar{\zeta}_t = \sum_{i=-\infty}^{t} \zeta_{i,t} \cdot \frac{I_{i,t}}{\sum_{j=-\infty}^{t} I_{j,t}} = \sum_{i=-\infty}^{t} \zeta_{i,t} \cdot \frac{I_{i,t}}{K_t} = \sum_{i=-\infty}^{t} \zeta_{i,t} \cdot S_{i,t} \tag{41}$$

where $S_{i,t}$ is the share of investment at time i in the capital stock at time t. We see then that $\bar{\zeta}_t$ is a weighted average of all individual 'capital productivities' of the separate vintages with the investment shares in the total capital stock (K_t) as weights. Note that when ρ_p is equal to 1, i.e. the ex-post elasticity of substitution is equal to 0.5, then (41) provides the 'exact' value of the aggregate capital productivity. When ρ_p is not equal to 1, we will assume that the average capital productivity (π_t) can be obtained as:

$$\pi_t = \{\bar{\zeta}_t\}^{1/\rho_p} \tag{42}$$

11 Note that when producers would have myopic foresight within this intertemporal setting (it should be admitted that this is a somewhat illogical supposition), equation (38) is consistent with the condition from the CRAPP model (see van Zon (1993), p. 9), that :

$$\nu_{i,t} = \{\nu_{t,t}\}^{\frac{\sigma_p}{\sigma_a}} \cdot (A^p_{i,t})^{\sigma_p} \cdot (A^a_{t,t})^{-\sigma_p}$$

Note that (42) implies that for ρ_p not equal to 1 the aggregate productivity of capital is obtained as a 'CES average' of the individual capital productivities at the vintage level, since (41) and (42) taken together imply :

$$\pi_t = \left\{ \sum_{i=-\infty}^{t} S_{i,t} \cdot \left(\frac{1}{\kappa_{i,t}} \right)^{\rho_p} \right\}^{1/\rho_p} \tag{43}$$

Equation (43) provides another (but 'tiny') approximation which is needed to define the RUM model. Assuming that (43) is indeed a good approximation of the arithmetical average of capital productivity [12] we can use (40) and (41) to obtain :

$$\bar{\zeta}_t = \sum_{i=-\infty}^{t} \frac{S_{i,t}}{B_{i,t}^p} - (\xi_t)^{-\rho_p} \cdot \sum_{i=-\infty}^{t} S_{i,t} \cdot \frac{(A_{i,t}^p)^{\sigma_p}}{B_{i,t}^p} \tag{44}$$

Equation (44) can be redefined as :

$$\bar{\zeta}_t = T_{1,t} - (\xi_t)^{-\rho_p} \cdot T_{2,t} \tag{45}$$

Note that in the absence of disembodied technical change, the two ways in which $T_{1,t}$ and $T_{2,t}$ depend on time are first through $S_{i,t}$ and secondly through the upper limits of the respective summations. For $T_{1,t}$ we conclude therefore that its value must be equal to $T_{1,t-1}$ except for the fact that the overall weight of already existing vintages in the determination of $T_{1,t}$ must have decreased when gross investment is positive, while on the other hand the relative weights $S_{i,t} / S_{j,t}$ for i,j<t are not changed at all (c.f. (41) and (4)). Therefore the transition from t-1 to t implies that the weight of existing machinery (i.e. the machinery installed up to and including time t-1) in the determination of the average value of capital productivity at time t has become $(1-\delta) \cdot K_{t-1}/K_t$, whereas the weight of the capital productivity of the new vintage in aggregate capital productivity is equal to $I_{t,t}/K_t$. A similar reasoning holds for the change in the value of $T_{2,t}$. We therefore have :

12 We will come back to this issue later on in the form of some illustrative simulations.

$$T_{1,t} = T_{1,t-1} \cdot \frac{(1-\delta)\cdot K_{t-1}}{K_t} + \left(\frac{1}{B^p_{t,t}}\right)\cdot\frac{I_{t,t}}{K_t} \tag{46}$$

$$T_{2,t} = T_{2,t-1} \cdot \frac{(1-\delta)\cdot K_{t-1}}{K_t} + \left(\frac{(A^p_{t,t})^{\sigma_p}}{B^p_{t,t}}\right)\cdot\frac{I_{t,t}}{K_t}$$

$$K_t = (1-\delta)\cdot K_{t-1} + I_{t,t}$$

$$\pi_t = \left\{T_{1,t} - \xi_t^{-\rho_p}\cdot T_{2,t}\right\}^{1/\rho_p}$$

With regard to disembodied technical change, it should be noted that (by assumption) it affects existing vintages only. Moreover disembodied technical change affects those vintages to the same extent. Hence, disembodied technical change can be introduced into the model quite easily by changing (46) into: [13]

$$T_{1,t} = T_{1,t-1} \cdot \frac{(1-\delta)\cdot K_{t-1}}{K_t}\cdot\left(\frac{1}{1+\gamma_I}\right)^{-\rho_p} + \left(\frac{1}{B^p_{t,t}}\right)\cdot\frac{I_{t,t}}{K_t} \tag{47}$$

$$T_{2,t} = T_{2,t-1} \cdot \frac{(1-\delta)\cdot K_{t-1}}{K_t}\cdot\left(\frac{(1+\gamma_n)^{\sigma_p}}{1+\gamma_I}\right)^{-\rho_p} + \left(\frac{(A^p_{t,t})^{\sigma_p}}{B^p_{t,t}}\right)\cdot\frac{I_{t,t}}{K_t}$$

$$K_t = (1-\delta)\cdot K_{t-1} + I_{t,t}$$

$$\pi_t = \left\{T_{1,t} - \xi_t^{-\rho_p}\cdot T_{2,t}\right\}^{1/\rho_p}$$

Equation (47) shows that the capital productivity 'book-keeping' of an infinitely large family of vintages can be reduced to a fairly small set of equations. [14] Moreover, equation (47) shows that the value of aggregate capital productivity can be obtained by means of a (time-) recursive update of its composing terms, rather than by explicitly obtaining it from the underlying individual vintages.

13 This way of handling disembodied technical change follows directly from the fact that an expression $X_t = X_0 \cdot (1+x)^{\beta\cdot t}$ can be written as : $X_t = X_{t-1}\cdot(1+x)^{\beta}$. Hence, X_t can be obtained by 'updating' X_{t-1} by means of the factor $(1+x)^{\beta}$. See also equation (28).

14 Note that a related approach is described in Eigenraam (1987), although the link between productivity aggregates and the development of the capital stock is less direct there than in the RUM model.

With regard to the determination of the aggregate labour / capital ratio, we can use a similar approach. Defining $\theta_{i,t} = \nu_{i,t}/\kappa_{i,t}$, the aggregate labour capital ratio ($\bar{\theta}_t$) can (implicitly) be written as :

$$\bar{\theta}_t^{\rho_p} = \sum_{i=-\infty}^{t} S_{i,t} \cdot (\theta_{i,t})^{\rho_p} = \sum_{i=-\infty}^{t} S_{i,t} \cdot (\nu_{i,t})^{\rho_p} \cdot \zeta_{i,t} = \sum_{i=-\infty}^{t} \frac{S_{i,t} \cdot (\nu_{i,t})^{\rho_p}}{B_{i,t}^p} - \sum_{i=-\infty}^{t} \frac{A_{i,t}^p}{B_{i,t}^p} \cdot S_{i,t} \tag{48}$$

$$= \xi_t^{\rho_p} \cdot \sum_{i=-\infty}^{t} S_{i,t} \cdot \frac{(A_{i,t}^p)^{\frac{\rho_p}{1+\rho_p}}}{B_{i,t}^p} - \sum_{i=-\infty}^{t} \frac{A_{i,t}^p}{B_{i,t}^p} \cdot S_{i,t} = \xi_t^{\rho_p} \cdot T_{3,t} - T_{4,t}$$

where we have used equation (40). Again, the terms $T_{3,t}$ and $T_{4,t}$ can be obtained by means of a recursive update mechanism. Introducing disembodied technical change into (48), we immediately obtain :

$$T_{3,t} = T_{3,t-1} \cdot \frac{(1-\delta) \cdot K_{t-1}}{K_t} \cdot \left(\frac{(1+\gamma_n)^{\frac{\rho_p}{1+\rho_p}}}{1+\gamma_I} \right)^{-\rho_p} + \frac{A_{t,t}^{p\,\frac{\rho_p}{1+\rho_p}}}{B_{t,t}^p} \cdot \frac{I_{t,t}}{K_t} \tag{49}$$

$$T_{4,t} = T_{4,t-1} \cdot \frac{(1-\delta) \cdot K_{t-1}}{K_t} \cdot \left(\frac{1+\gamma_n}{1+\gamma_I} \right)^{-\rho_p} + \frac{A_t^p}{B_t^p} \cdot \frac{I_{t,t}}{K_t}$$

Equations (46),(48) and (49) can be used to obtain total capacity labour demand and total production capacity as :

$$N_t = \bar{\theta}_t \cdot K_t = \left\{ \xi_t^{\rho_p} \cdot T_{3,t} - T_{4,t} \right\}^{1/\rho_p} \cdot K_t \tag{50}$$

$$X_t = \pi_t \cdot K_t = \left\{ T_{1,t} - \xi_t^{-\rho_p} \cdot T_{2,t} \right\}^{1/\rho_p} \cdot K_t$$

Of course, the replacement of a full putty-semi-putty vintage model by its RUM representation has its price. **First** of all, for ex-post elasticities of substitution not equal to 0.5, the RUM model is only an approximation of the full vintage model (although a good one), while **secondly** the terms $T_{1,t}$ through $T_{4,t}$ are recursively defined, and hence need to be initialised. However, in a growing economy it follows that the term $(1-\delta) \cdot K_{t-1}/K_t = (1-\delta)/(1+g)$ (where g is the rate of growth of the capital stock) is smaller than one, and it is easily seen that the influence of any initial value of the individual terms $T_{1,t}$ through $T_{4,t}$ tends to diminish over time. Moreover, this happens more rapidly when the rate of technical decay is high or when the rate of growth of the economy is high. Nonetheless, for short sample periods, initial values $T_{1,0}..T_{4,0}$ will have to be 'estimated' next to the other parameters of the production structure.

The logic of the model is now as follows. First, the technological characteristics of the ex-ante function together with expectations regarding factor prices and technological change, determine the initial technique on the 'newest' ex-post production function. This in turn determines the reference value for marginal labour productivity to be used for the allocation of labour to existing equipment. Thus we obtain the level of output on existing machinery for a given value of the stock of existing capital, as well as the associated amount of labour. Then we obtain the amount of output to be produced on the newest equipment as the difference between the total amount of output required, and the amount of output to be produced on existing equipment. After that, the required amount of investment as well as the associated amount of labour on new equipment can be obtained from the optimum values of the factor productivities on new equipment.

In the following section we will provide some simulation results using this particular model, without going into the problem of the econometric estimation of the RUM model yet. As a reference model, we will also use the full putty-semi-putty vintage model which was put forward in this section, with the proviso that we only do the 'book-keeping' for the last hundred vintages installed. [15] The vintage reference model will further be referred to as 'VRM'.

3 Some Illustrative Simulations with RUM

3.1 Introduction

In this section we present the outcomes of a number of experiments we have conducted with the RUM model. First, however, we present the values of the parameters and the initial values of the model variables which we have used in order to obtain a base-run. The values are listed in table 3.1 below. In this table G(Q) stands for the proportional rate of growth of Q, i.e. $G(Q) = Q_t / Q_{t-1} - 1$. w_0 is the initial value of the wage rate. The average rate of growth of nominal wages is assumed to be 2.5 percent, while the rate of growth of output is assumed to be 3 percent. X_0 is the initial value of output.

15 Note that the assumption of a 10 percent technical decay per annum implies that we can safely ignore the vintages installed before and up to time t-99.

Table 3.1 Parameter Values and Initial Values

Parameter / Variable	Value	Parameter / Variable	Value
σ_a	0.667	G(w)	0.025
σ_p	0.25	w_0	1
A_0	0.5	u	1
B0	0.5	X_0	500
μ_n	0.025	G(X)	0.03
μ_I	0.00	γ_n	0.015
δ	0.1	γ_I	0.00

Using the parameter values mentioned above, we ran a number of different experiments. Experiment 0 is called the base-run. The base-run is obtained using a planning period of infinite length. The most important variables are depicted in a number of figures associated with each experiment. Moreover, the names of these variables differ in accordance with the experiments : a common ’root’ is followed by a string which denotes the experiments in question. More in particular, experiment number ’x’ has extension ’_x’. Apart from these ’experimental extensions’, the series names also provide information with regard to the type of model which generated them. Series with the extension ’_RUM’ indicate that these are generated using the RUM model, while the extension ’_VIN’ is associated with series generated using the VRM (putty-semi-putty vintage reference model.

In experiment 1 we let the planning horizon fall from an infinite value to zero (pure myopic foresight) from period 175, while it regains its infinite value again from period 225. This run therefore shows a zero planning period during the period 175-225 within a fully intertemporal framework. Although this is somewhat of a ’contradictio in terminis’, this run can be used to illustrate the working of a lengthening of the planning period in terms of the implied gap between the tangential technique on the one hand and the initial technique on the ex-post iso-quant (which is defined by the tangential technique) on the other.

In experiment 2 we let the rate of growth of nominal wages increase to 3.5 percent during the period 175-225. This will make labour a more expensive factor of production, both in the short run and in the long run.

In experiment 3 we let the rate of embodied labour augmenting technical change μ_n be equal to 3.5 percent during the period 175-225 only, whereas it is equal to 2.5 percent for the rest of the period.

In experiment 4 we increase the rate of disembodied labour augmenting technical change by 1.5 percentage points, i.e. γ_n will have a value of 0.03 instead of 0.015, but again only for the period 175-225. Experiments 3 and 4 will provide insight into the similarities and differences between embodied and disembodied technical change both in the RUM case and in the VRM case.

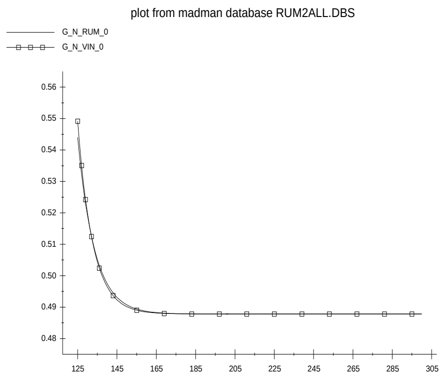
In experiment 5, we repeat experiment 4 but at the same time we change the ex-post elasticity of substitution from its default value of 0.25 to a value of 0.5. As one may recall, the latter value implies that the RUM implementation of the VRM should generate exactly the same results as the VRM itself. Moreover, this experiment will provide insight into the consequences of a change in the elasticity of substitution ex-post for the growth of employment and the like.

In experiment 6, we let the rate of growth of nominal wages fluctuate in a uniformly distributed random fashion in between 0 and 5 percent. Thus, an average rate of growth of 2.5 percent is obtained, which is the same value as the rate of growth of nominal wages used in the other experiments.

We now proceed with a short description and evaluation of the experiments mentioned above.

3.2 Experiment 0 : the Base-run

In figures 2 and 3 below, we show the rate of growth of capacity labour demand and of the aggregate labour / capital ratios as they are obtained from both the VRM and the RUM model with an infinite planning horizon. It is easily checked that, apart from some initialisation errors during the beginning of the simulation period (which runs from period 101 up to period 300), the RUM model outcomes closely resemble the VRM outcomes. These initialisation errors become somewhat more apparent in figure 3, although the differences in the growth rates are very moderate indeed.



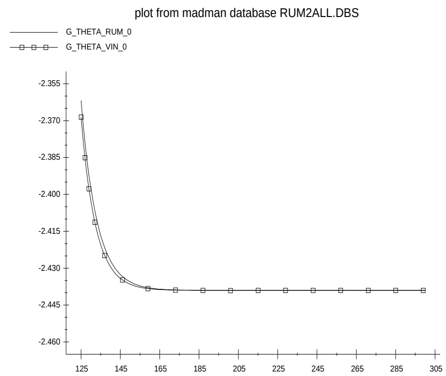
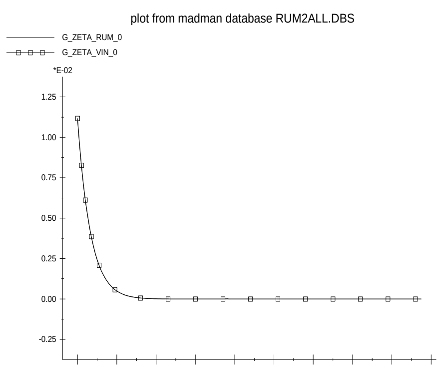


Fig 2. Aggregate Capacity Labour Demand (% Growth)

Fig 3. Aggregate Labour / Capital Ratios (% Growth)

A similar conclusion holds with respect to the capital coefficient of production which is depicted in figure 4. Note that the positive growth of the capital labour ratio is a consequence of the rise in relative wages.



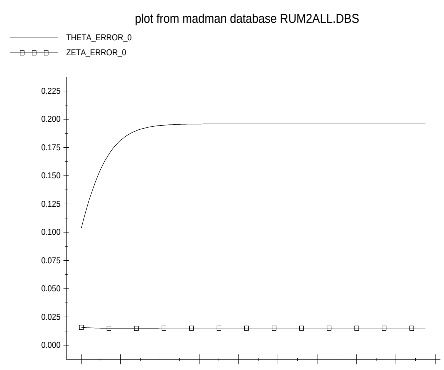


Fig 4. Aggregate Capital Productivity (% Growth)

Fig 5. Relative Percentage Errors Aggregate Labour / Capital Ratios and Capital Productivity

In figure 5, we have depicted the percentage errors in the aggregate labour / capital ratio and the aggregate capital productivity of the RUM model relative to the VRM model. We see that the percentage error in the value of capital productivity in particular is of negligible importance : it is less than 0.025 percent, while the error in the aggregate labour/capital ratio is somewhat larger, but still less than 0.2 percent. We conclude that aggregate values are generated quite well, although the elasticity of substitution ex-post is equal to 0.25 rather than 0.5.[16]

In figure 6 below, we have depicted the rate of growth of the labour coefficient which is part of the tangential technique and of the initial labour coefficient on the ex-post iso-quant which is identified by that particular tangential technique. In the long run both techniques grow with the same (negative) rate of growth, which is caused by the positive rates of embodied and disembodied labour augmenting technical change and of relative wages.

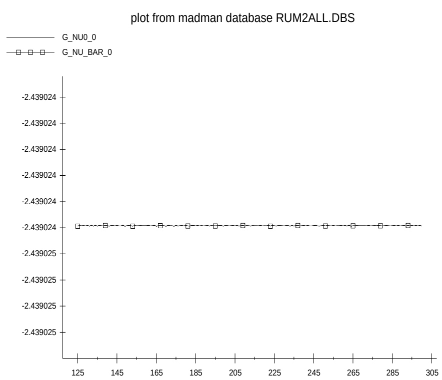


Fig 6. Tangential and Entry-Point Labour/Output Ratio (% Growth)

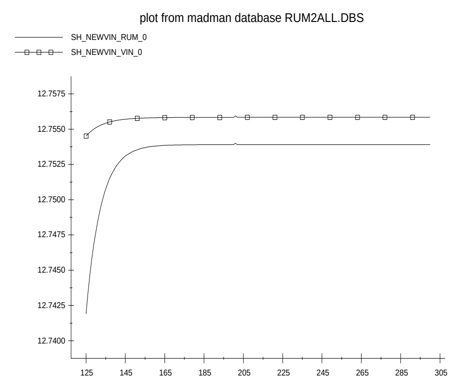


Fig 7. Production Shares Newest Vintage (%)

In figure 7 we present the production shares of the newest vintage in total capacity output. Since the investment time-series which is used in the RUM model is generated by the VRM model, the slight differences one observes in the production shares reflect the equally slight differences in marginal and aggregate capital productivities in both cases. Note too, that in the long run, production shares remain stable at about 12.75 percent.

3.3 Experiment 1: A Sudden Touch of Myopia

In this experiment we let the planning horizon fall from an infinite value to a zero value during the experimental period 175-225. Of course, in an intertemporal setting this does

16 The latter value of the elasticity of substitution ex-post should generate exact results, as will be shown in experiment 5.

not make much sense, but we use this experiment to show how a change in the length of the planning period may influence the composition of the capital stock in terms of old and new equipment as well as the technological characteristics of the capital stock.

In figures 8 and 9 below, we present the rates of growth of capacity labour demand and the rates of growth of the aggregate labour / capital ratios.

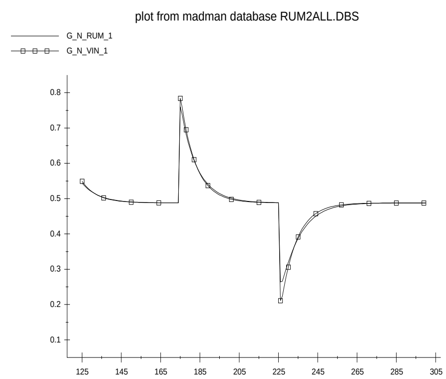


Fig 8. Aggregate Capacity Labour Demand (% Growth)

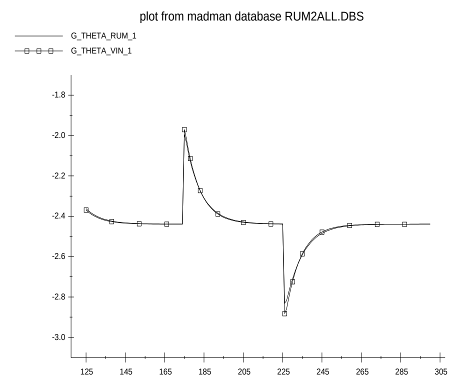


Fig 9. Aggregate Labour / Capital Ratios (% Growth)

From figure 8 it is clear that a fall in the planning period causes a rise in the demand for labour. The latter is also reflected by an increase in the aggregate labour intensity of production as is shown in figure 9. This result seems to be somewhat strange at first sight, since one would expect future gains in labour productivity due to disembodied technical change to have had a positive effect on capacity labour demand in general. However, one should note that nominal wage growth more than offsets the rise in labour productivity due to disembodied technical change. Therefore, a fall in the length of the planning period also implies a fall in the relevant relative wage rate (c.f. (34)). Hence, the rate of growth of the initial labour coefficient associated with the newest vintage rises. From figure 10 it can be seen that the rate of growth of the aggregate capital productivity rises in order to compensate for the fall in aggregate labour productivity caused by the relative fall in wages. From figures 8 and 9 it is furthermore clear that the impact of a fall in the planning horizon is distributed over time. Of course, this is a consequence of the fact that the changes at the aggregate level are caused by changes at the vintage level, and that a change in the relative price of labour has its strongest effect on the new vintage, and a much less outspoken effect on existing vintages (depending on the relative values of the elasticities of substitution ex-ante and ex-post).

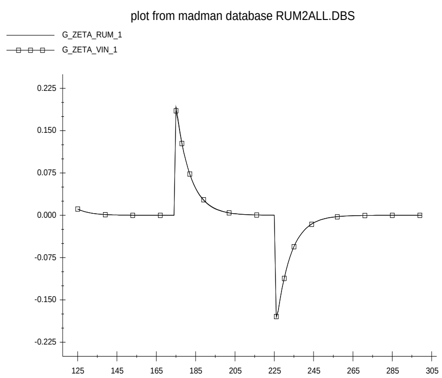


Fig 10. Aggregate Capital Productivity (% Growth)

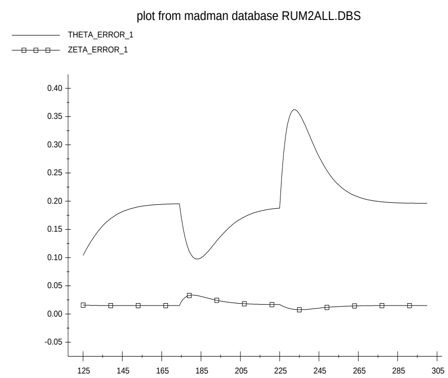
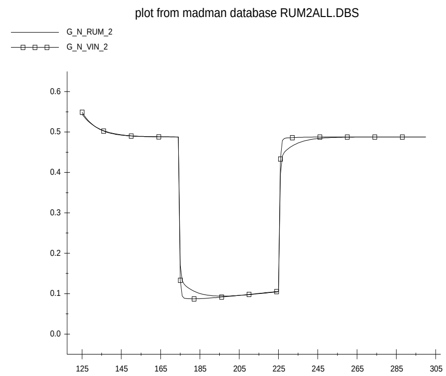


Fig 11. Relative Percentage Errors Aggregate Labour / Capital Ratios and Capital Productivity

From figure 11 above, we conclude that relative errors between RUM and the full putty-semi-putty vintage model become slightly less during the experimental period, and worse again (as a kind of echo-effect) after the experimental period when the planning horizon becomes infinite again. However, this is a temporary phenomenon, while moreover relative errors are still quite small.

3.4 Experiment 2: An Increase in the Rate of Growth of Nominal Wages

In this experiment we let the growth of nominal wages be equal to 3.5 percent instead of 2.5 percent during the experimental period. In figures 12 and 13 below, we show the effects this has on the growth rates of aggregate capacity labour demand and on the growth rates of the aggregate labour / capital ratios.



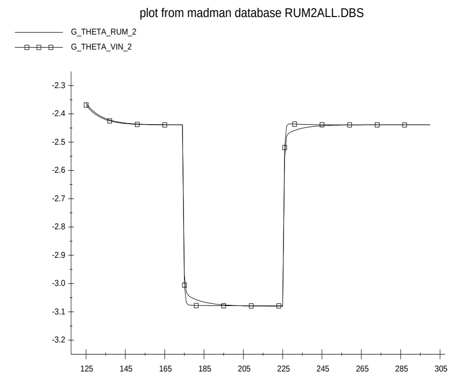
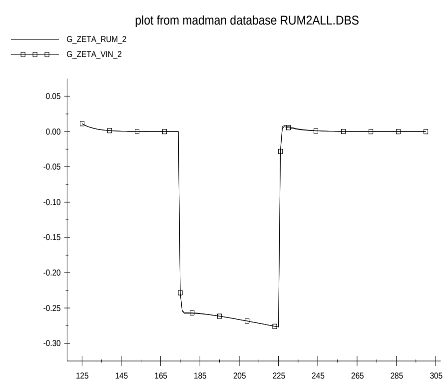


Fig 12. Aggregate Capacity Labour Demand (% Growth)

Fig 13. Aggregate Labour / Capital Ratios (% Growth)

Note that the rise in the growth rate of nominal wages affects all vintages at once, both old ones and the new one. Hence, the drop in the growth rate of capacity labour demand and the labour intensity of production is of approximately equal size during the entire experimental period. This result is principally different from the previous result, where the main impact of a change in planning periods on aggregate capacity labour demand comes through the newest vintage only. In this particular case both the tangential technique is affected as well as the techniques on all individual ex post production functions.



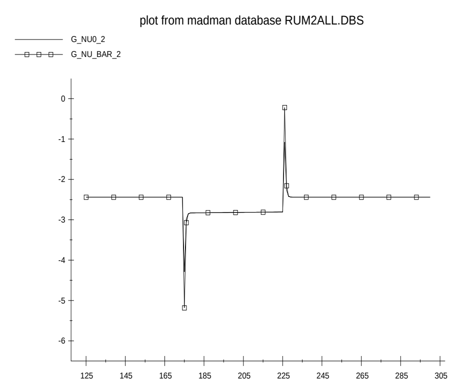


Fig 14. Aggregate Capital Productivity (% Growth)

Fig 15. Tangential and Entry-Point Labour / Output Ratio (% Growth)

In figure 14 we see that aggregate capital productivity growth is negatively affected by the increase in the growth rate of nominal wages, as one would expect. Note too that a slight vintage effect is visible here. In figure 15 we have depicted the rates of growth of the labour coefficients of the tangential technique and of the initial technique. Note that both at the start and at the end of the experimental periods the rates of growth are much larger than during the experimental period. This is caused by the sudden jump in the rate of growth of nominal wages in periods 175 and 225. This jump causes a relatively large adjustment of employment, since the tangential technique in particular is shifted towards a more capital intensive production technology. The reason is that, by construction, the higher rate of growth of nominal wages is expected to persist, thus raising the 'present value price' of labour.

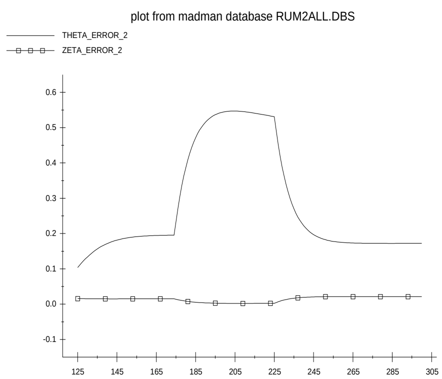


Fig 16. Relative Percentage Errors Aggregate Labour / Capital Ratio and Capital Productivity

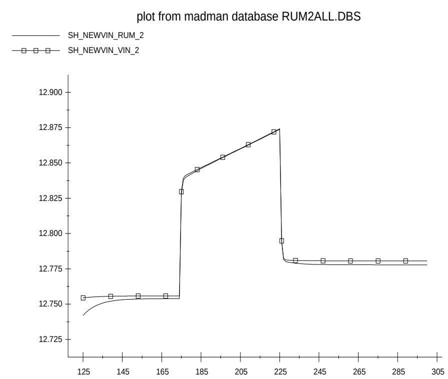


Fig 17. Production Shares Newest Vintage (%)

In figures 16 and 17, we present the relative percentage error and the production share of the newest vintage in total capacity output. Again, it can be seen that relative errors are affected during the experimental period, but only slightly so in case of aggregate capital productivity and to a somewhat larger extent in case of the aggregate labour / capital ratios. Nonetheless, relative errors are still quite small. From figure 17 it can be seen that the share of the newest vintage in total production capacity is positively affected by an increase in the growth rate of nominal wages. The reason for this is that the higher growth of wages forces marginal productivity on the newest vintage up, while the first order optimality conditions imply that marginal labour productivity on older vintages need to be adjusted towards this higher level of marginal labour productivity. The only way in which this can be brought about is by lowering the amount of labour allocated to existing vintages, and consequently, the amount of output produced using old vintages only falls. Hence, the size of the newest vintage, which fills the gap between total output required and total output produced on old vintages, has to rise in order to match the fall in the level of output on old vintages.

3.5 Experiment 3: A Sudden Rise in the Rate of Embodied Labour Augmenting Technical Change

In figures 18 and 19 below, we present the rates of growth of aggregate capacity labour demand and of the aggregate labour/capital ratios. In figure 18 in particular we see the vintage effect at work, in the sense that the implied fall in the labour intensity of production (where labour is measured in physical units of course) happens gradually over time. However, the growth of the aggregate labour/capital ratios is affected more or less evenly during the entire experimental period.

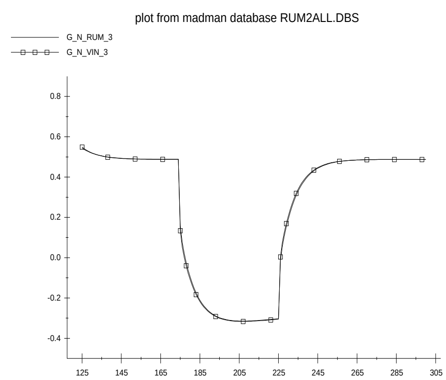


Fig 18. Aggregate Capacity Labour Demand (% Growth)

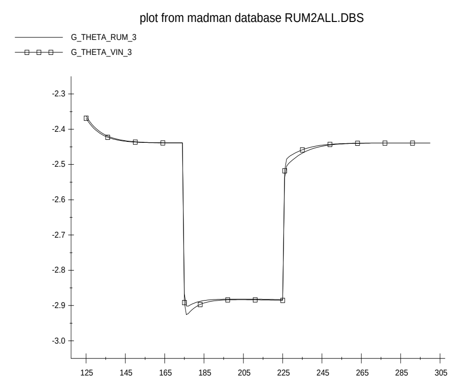


Fig 19. Aggregate Labour/Capital Ratios (% Growth)

Since the rate of growth of the capital stock is by definition equal to the rate of growth of capacity labour demand less the rate of growth of the aggregate labour/capital ratio, it follows that the more or less sudden change in the rate of growth of the aggregate labour/capital ratio is caused by two movements over time which have similar time-profiles but different average levels.

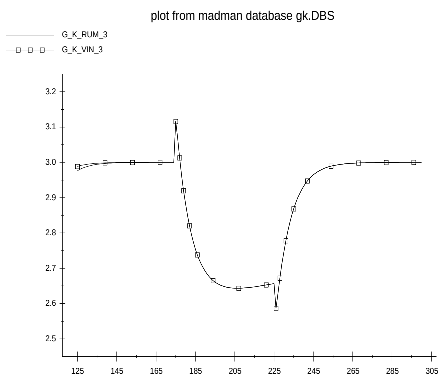


Fig 20. Capital Stocks (% Growth)

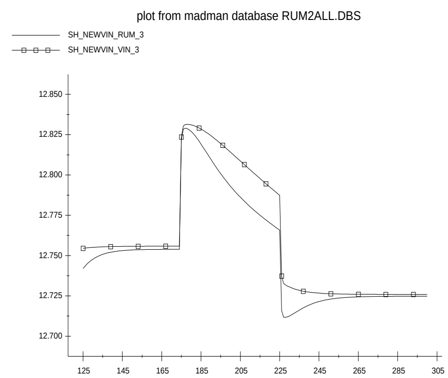


Fig 21. Production Shares Newest Vintage (%)

A comparison of figures 18 and 20 shows that this is indeed the case. Moreover, looking more closely at figure 20, one notices first a one time acceleration in the growth of the capital stock, immediately followed by a more or less permanent deceleration. The opposite happens when the experimental period ends, and the rate of embodied labour augmenting technological progress changes again to its original value of 2.5 percent per annum. This reversal of signs is caused by the rise in the rate of scrapping which is induced by the rise in the marginal productivity of labour on the newest vintage, which is in turn caused by the rise in the rate of embodied labour augmenting technological change. This sequence of events is also in part reflected by figure 21, where the production share of the newest vintage has been depicted. We observe a sharp initial rise (although rather limited in absolute terms), which is then followed by a gradual decline over time of the share of the newest vintage. Why does this share decline after its initial rise ? The answer is that the capital stock, while growing older, will consist more and more of vintages with a relatively high level of embodied technical change, which raises the average (initial) marginal productivity of existing vintages. Therefore, matching the marginal labour productivity on the newest vintage by allocating less labour to existing vintages becomes more easy on average. Hence, there is a tendency for production on old vintages to fall less rapidly than in the beginning of the experimental period.

3.6 Experiment 4: A Sudden Rise in the Rate of Labour Augmenting Disembodied Technical Change

In this experiment we have suddenly raised the rate of disembodied labour augmenting technical change from a value of 1.5 percent to a value of 3 percent in the experimental period 175-225. From period 226, the original value of the rate of disembodied labour augmenting technical change is restored again. In figures 22 and 23, below we show the rates of growth of aggregate capacity labour demand and of the aggregate labour / capital ratios.

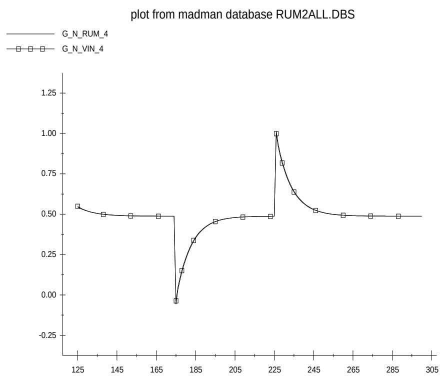


Fig 22. Aggregate Capacity Labour Demand (% Growth)

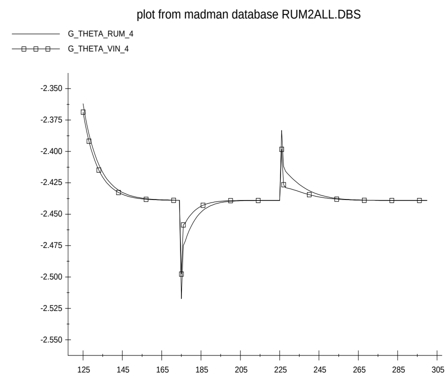


Fig 23. Aggregate Labour / Capital Ratios (% Growth)

Comparing figures 22 and 18 we notice that the time-profile of the employment effects of a change in the rate of embodied technical change is totally different from the corresponding time-profile of the effects of a similar change in the rate of disembodied technical change. While in the case of embodied technical change, the impact reaches its maximum in a more or less gradual way, we notice that in the case of disembodied technical change, the impact falls from its maximum level attained directly after the shock to a zero value after about 30 periods. Hence, in the long run the impact of disembodied technical change on employment growth is not visible (a change in the rate of disembodied technical change causes a level-jump in employment), whereas a change in the rate of embodied technical change has an effect on the growth rate of employment, even in the steady state.

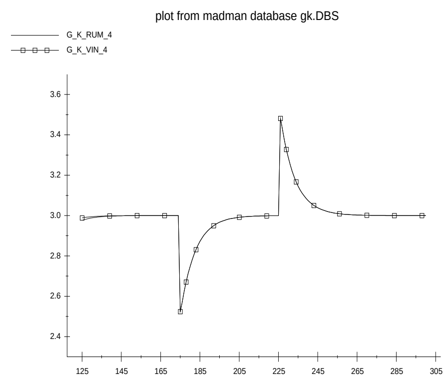


Fig 24. Capital Stocks (% Growth)

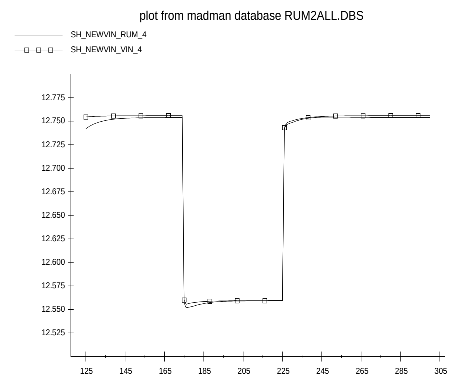


Fig 25. Production Shares Newest Vintage (%)

Note that in figures 24 and 25, the rise in the rate of disembodied technical change makes existing vintages more productive relative to the newest one. Hence, the marginal labour productivity gap between old vintages and the newest vintage is diminished somewhat, which Leads to a decrease in the rate of scrapping and hence to a fall in the production share of the newest vintage. Because all existing vintages are affected by the rise in the rate of disembodied technical change to the same extent, there is no vintage effect to be observed in the time profile of the production share of the newest vintage, since the productivity gap between new investment and old capital is constant as long as there is a constant difference between the rate of embodied technical change and the rate of disembodied technical change. The latter is clearly the case during the entire experimental period except for its beginning and its end.

3.7 Experiment 5: RUM as an Exact Representation of the Full Putty-Semi-Putty Vintage Production Model

In this experiment we have rerun experiment 4 with a value of the elasticity of substitution ex-post equal to 0.5. As one may recall, the latter value implies that the aggregate values of capital productivity and of the capital / labour ratio which were approximated by CES averages of the corresponding concepts at the vintage level, are now exact representations of these concepts. Consequently, both the RUM-model and the full putty-semi-putty model should generate identical results. However, this is not what happens, due to initialisation errors, which disappear almost totally in just a few years. As usual, the growth rate of capacity labour demand and of the aggregate labour / capital ratio is depicted in figures 26 and 27 below.

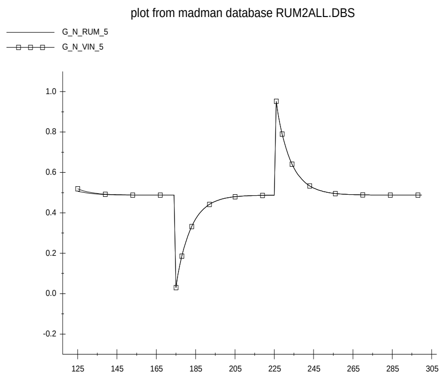


Fig 26. Aggregate Capacity Labour Demand (% Growth)

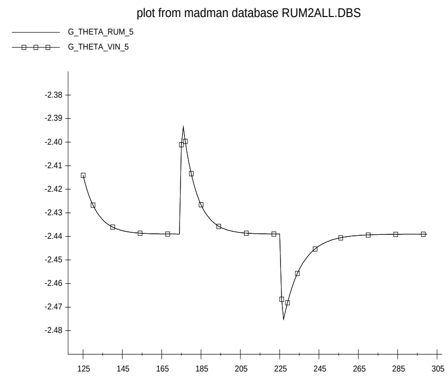


Fig 27. Aggregate Labour / Capital Ratios (% Growth)

As stated above, apart from some initialisation errors, the differences between the RUM outcomes and the VRM outcomes are negligible in size. This can be observed more clearly in figure 28, where we have depicted the relative percentage error between RUM and the VRM model.

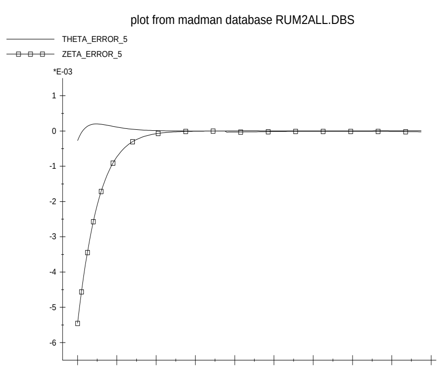


Fig 28. Relative Percentage Errors Aggregate Labour / Capital Ratio and Capital Productivity

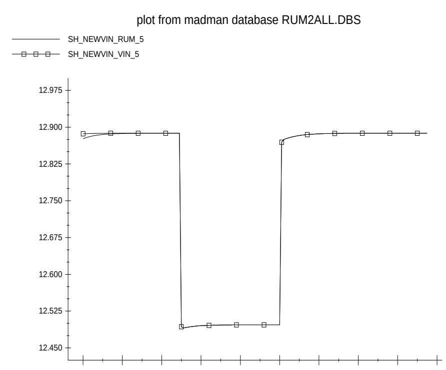


Fig 29. Production Shares Newest Vintage (%)

As can be seen from figure 28 the relative errors are indeed quite small. RUM under-estimates the VRM results by about 0.005 percent at first, but even this small difference disappears completely over time. Identical results can also be observed in the case of the production share of the newest vintage for both models. However, there is a difference to be noted with respect to experiment 4. [17] While the rate of growth of capacity labour demand is very similar indeed in both experiments 4 and 5, the same can not be said with respect to the rate of growth of the aggregate labour / capital ratio. First of all, the effects in experiment 5 are only about half the size of the effects in experiment 4 in absolute terms, while moreover they are reversed in sign. The reason is of course that a rise in the elasticity of substitution ex-post implies that the fall in the effective price of labour-services (due to the rise in the rate of disembodied labour augmenting technical change) leads to larger substitution effects on the existing vintages, which become relatively more labour intensive than in experiment 4. This is not only reflected by the relative rise in the aggregate labour / capital ratios, but also by the relative fall in the production share of the newest vintage. For the relative increase in the allocation of labour to existing vintages has the effect that total output associated with those vintages increases too. This can easily be verified by comparing figures 25 and 29.

3.8 Experiment 6: Random Wage Growth

Up to now we have concentrated on longer term responses of both the RUM-model and the VRM model to shocks in the different rates of technological change, as well as to shocks in the rate of growth of nominal wages. We have observed that the adjustment paths, which are the result of those shocks, are first of all very similar indeed for both types of models, and secondly fairly smooth over time. In this experiment we will look into the question whether or not the RUM model is able to follow short term fluctuations in wages as nicely as it is able to follow more permanent changes in economic circumstances. To this end, we have generated random growth rates in nominal wages which are uniformly distributed over the range 0-5 percent. Hence, the average rate of growth of nominal wages is still equal to 2.5 percent. In figures 30 and 31 below, we present the growth rate of capacity labour demand and of the aggregate labour / capital ratio. Note that in this experiment the old value of the elasticity of substitution ex-post (equal to 0.25) has been used. Still the experiments show that both time-series are virtually identical.

17 As one may recall, experiment 4 is identical to experiment 5, except for the value of the elasticity of substitution ex-post which has a value of 0.25 as compared to 0.5. Hence experiment 5 represents a rise in the elasticity of substitution ex-post relative to experiment 4.

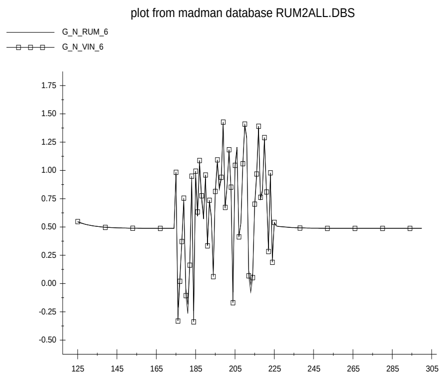


Fig 30. Aggregate Capacity Labour Demand (% Growth)

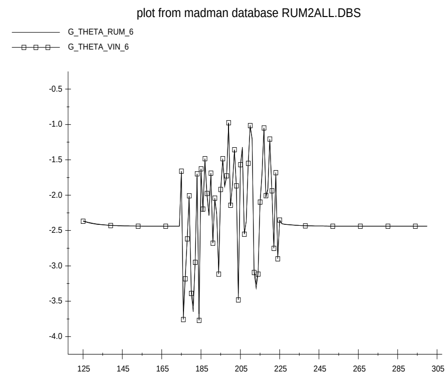


Fig 31. Aggregate Labour / Capital Ratios (% Growth)

Note too, from figures 32 and 33 below, that although relative errors increase somewhat during the experimental period, they are still less than 0.5 percent, with an average of about 0.25 percent for the aggregate labour / capital ratios and an average of less than 0.05 percent for the level of aggregate capital productivity. Even short term changes in investment are approximated rather well, as can be seen from the short term changes in the production share of the newest vintage.

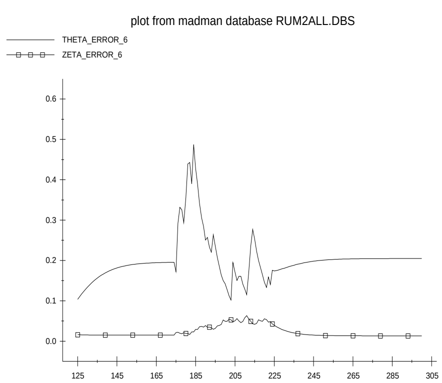


Fig 32. Relative Percentage Errors Aggregate Labour / Capital Ratio and Capital Productivity

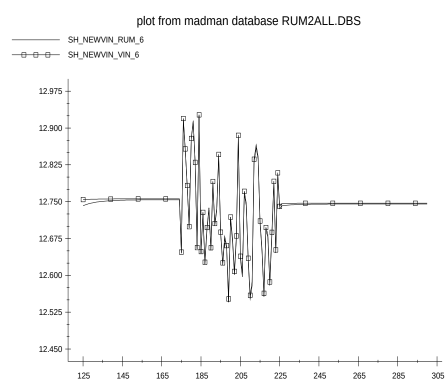


Fig 33. Production Shares Newest Vintage (%)

4 Summary and Conclusion

In this paper we have set out a fully intertemporal version of the CRAPP model which was presented in van Zon (1993). The name CRAPP stands for 'Continuous Recursive

Adjustment Putty-Putty' (model). We have combined a putty-semi-putty vintage setting, with limited but positive substitution possibilities ex-post, with a set of recursive adjustment / update rules with respect to average capital productivity and the average value of the labour / capital ratio. These rules describe the changes in the average technological characteristics of the capital stock in terms of the changes in the technological characteristics of both existing capital and new capital goods. In this way we are able to integrate the notions of embodied and disembodied technical change into a quasi-vintage model which is a very good approximation of a full putty-semi-putty vintage model, although it requires only a fraction of the computational overhead which is usually required by a full (putty-semi-putty) vintage model. The CRAPP model was defined for a situation of myopic behaviour by entrepreneurs.

Then came RUM as a logical descendant of CRAPP. RUM is short for Recursive Update Model, and it is an intertemporal version of the CRAPP model. In RUM, producers do not only choose a technique on some pre-existing iso-quant, but since they know that their substitution possibilities ex-post are smaller than the ones ex-ante, they also select entire ex-post iso-quants by choosing the appropriate initial technique in the ex-post iso-quant field, conditional on the requirement that all new ex-post iso-quants at some moment of time are 'enveloped' by the ex-ante iso-quant. The latter iso-quant shifts through the labour-capital space due to embodied technical change. Ex post iso-quants, once they are chosen, shift through that same space too, but this is due to disembodied technical change.

In this paper we have shown that an ex-post iso-quant is uniquely defined by the point of tangency of the ex-post iso-quant in question and the 'enveloping' ex-ante iso-quant. Hence, choosing an optimum production technique now comes down to choosing an optimum 'tangential technique', and an optimum entry-technique on the ex-post iso-quant which is uniquely defined by that 'tangential technique'. These choices are made in an intertemporal setting where we show that the allocation of labour among new and among existing vintages of capital are ruled by a putty-semi-putty equivalent of the Malcomson scrapping condition.

Using the RUM model as a representation of a full putty-semi-putty vintage model with linear homogeneous CES production functions both ex-ante and ex-post, we showed that it is not strictly necessary to engage in extensive vintage 'book-keeping' exercises. Rather, the application of a very limited set of recursive update rules with respect to the aggregate capital productivity and the aggregate labour / capital ratio in function of the characteristics of the new vintage to be installed, is sufficient to reproduce virtually all the relevant information generated by the full putty-semi-putty vintage model.

We illustrated the working of the RUM model in a number of experiments. The general conclusion one can draw from the experiments is that, apart from disturbances caused by initialisation errors, the RUM model and the full putty-semi-putty model are indeed almost perfect 'look-alikes' with respect to aggregate behaviour.[18] The experiments have shown that not only asymptotic behaviour of both models is very much the same, but also short term fluctuations in the full putty-semi-putty vintage model environment are captured almost perfectly by the RUM model. This goes for fluctuations in the growth of relative prices, but also for fluctuations due to technology shocks.

In addition, the experiments have shown that there is indeed a principal difference between the impact of disembodied technical change as compared to the impact of embodied technical change. First of all, disembodied technical change affects the entire capital stock at once, while embodied technical change only gradually affects the capital stock. Moreover, in the long run, only changes in the rate of embodied labour augmenting technical change have permanent growth effects.

In conclusion we may state then that the RUM model has the flavour of a full putty-semi-putty vintage model, and the experiments we have run all show that in practice it behaves as such a model. At the same time it consists of a very limited set of equations which uses the idea of a time-recursive update of aggregate capital productivity and the aggregate capital/labour ratio in function of the characteristics of new investment and (induced) changes in the characteristics of old equipment. Thus, the RUM model behaves as if it is a full vintage model, while at the same time it avoids tracing individual vintages during their (infinite) lifetime. Nonetheless the RUM model is able to generate exact results for an elasticity of substitution ex-post of 0.5, and nearly exact results for a value of the elasticity of substitution ex-post of 0.25.

Considering its performance during a number of different experiments, we may conclude that the RUM model can serve as a comprehensive and manageable alternative to the large computational and 'book-keeping' burden which a standard vintage model approach usually entails.

18 Note that with respect to marginal behaviour, the equations describing the installation of new capital goods are (by definition) completely identical.

Appendix A : Some Algebra

Equation (23) says :

$$q'_j = \sum_{i=j}^{\infty} \lambda^n_{j,i} \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot \frac{(1-\delta)^{i-j}}{Y_{j,i}} = \sum_{i=j}^{\infty} (p'_i - \lambda^x_i) \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot (1-\delta)^{i-j} \tag{A.1}$$

Hence :

$$q'_{j+1} = \sum_{i=j+1}^{\infty} (p'_i - \lambda^x_i) \cdot \frac{\partial f^{j,i}}{\partial \kappa_{j,i}} \cdot (1-\delta)^{i-j-1} \tag{A.2}$$

Since $q'_{j+1} = (1+\hat{q}_j) \cdot (1+r_t)^{-1} \cdot q'_j$ (where $\hat{q}$ is the rate of growth of q), it follows immediately that :

$$q'_j - (1-\delta) \cdot q'_{j+1} = q'_j \cdot \left(1 - \frac{(1-\delta) \cdot (1+\hat{q}_j)}{1+r_t}\right) = (p'_j - \lambda^x_j) \cdot \frac{\partial f^{j,j}}{\partial \kappa_{j,j}} \tag{A.3}$$

Multiplication of (A.3) by $\kappa_{j,j}$ and using the Euler equation, gives :

$$q'_j \cdot \kappa_{j,j} \cdot \left(1 - \frac{(1-\delta) \cdot (1+\hat{q}_j)}{1+r_t}\right) = (p'_j - \lambda^x_j) \cdot \left(1 - \frac{\partial f^{j,j}}{\partial \nu_{j,j}} \cdot \nu_{j,j}\right) = p'_j - \lambda^x_j - w'_j \cdot \nu_{j,j} \quad \Rightarrow \tag{A.4}$$

$$p'_j - \lambda^x_j = q'_j \cdot \kappa_{j,j} \cdot \left(1 - \frac{(1-\delta) \cdot (1+\hat{q}_j)}{1+r_t}\right) + w'_j \cdot \nu_{j,j}$$

where we have substituted equations (18.B) and (21) into the expression for the marginal productivity of labour in the second part of (A.4). Note that (A.4) reproduces the familiar expression for the user cost of capital per period as the sum of the rate of technical decay and the rate of interest less the proportionate capital gains on existing machinery. Note too, that $p'_i - \lambda^x_i$ measures the user cost of both labour and capital per unit of output on the newest equipment.

Appendix B : The RUM Model Listing

```
{
     November RUM Model in MESS-Language
  MESS (C) Menhir Software Group
            Ringweg 46
            6271 AK Gulpen
            the Netherlands

}
{   RUM stands for Recursive Update Model. Putty-Semi-Putty Model with variable
    distribution coefficients in CES function and recursive updating of average capi-
    tal productivity and labour intensity on old equipment in function of  changes in
    characteristics of new equipment

  types of technological change :
   embodied labour augmenting at     rate  mu_n
   embodied capital augmenting at    rate mu_i
   disembodied labour augmenting at  rate gamma_n
   disembodied capital augmenting at rate gamma_i
   changes in the respective rates of labour saving technical change which represent a
   'technology shock' are labelled d_mu_n and d_gamma_n, respectively
   we perform 6 different experiments.we simulate from period 101 upto period
   300. a shock is applied only during the 'experimental period' 175-225
0 ..    baserun : infinite horizon
1 ..    as 0 but planning period becomes zero (myopic foresight) in experimental
        period
2 ..    as 0 but growth of nominal wages is increased by 1 percent
3 ..    as 0 but rate of embodied labour augmenting technical change is      increased
        by 1.5 percent during experimental period only
4 ..    as 0 but rate of disembodied labour augmenting technical change is increased
        by 1.5 percent during experimental period
5 ..    as 4 but with substitution elasticity ex post equal to 0.5
6 ..    as 0 but random wage growth with mean 2.5 percent. uniformly distributed
        rates of growth in between 0 and 5 percent

        rrandom is random number generator (uniformly distributed). maximum value
        is equal to 1. Before first draw initialise random generator in such a way that
        for all simulations the same sequence is generated => define function rrandom
        in separate Turbo-Pascal Unit
}
```

```
{THE MODEL}
 wagerate = wagerate(-1)*
             (1+if_else_(and_(ge_(time,start_exp),
                          le_(time,stop_exp)),
                      if_else_(experiment2,
                            gw+d_gw,
                            if_else_(experiment6,
                                  if_else_(eq_(iter,0),
                                        2*rrandom*gw,
                                        w_hat),
                                  gw)
                            ),
                      gw
                     )
             ),


    {first some series for both models}

w_hat = wagerate / wagerate(-1)-1,

   {note cum_emb(..) is the cumulative impact of embodied technical change for a
   vintage installed at time time at rate mu_n in normal circumstances and rate mu_n
   + d_mu_n when experiment 3 is active and when the year of installation is within
   the experimental period
}

a_exante = a0 * p_(1.04 / (1+mu_n),0-rho_exante*105)*
             p_(cum_emb(time,
                     baseyear,
                     mu_n,
                     mu_n+if_else_(experiment3,d_mu_n,0),
                     start_exp,
                     stop_exp),
               -rho_exante),



b_exante = b0 * p_(1+mu_i,-rho_exante*(time-baseyear)),

   {first calculate tangential technique in myopic foresight case, and use this tech-
   nique to obtain a first estimate of the unit costs on the new vintage. The optimum
   tangential technique at myopic foresight depends on the current user cost of
   capital (ucc) and the current wagerate (wagerate)
}

q_hat = q / q(-1)-1,
qacac = q * (1 - (1-delta)*(1+q_hat) / (1+r)),
h_myopic = p_((wagerate / qacac)* (b_exante / a_exante), -1 / (1+rho_exante)),
```

```
    { 0 is youngest vintage }

kappa_myopic = p_(a_exante * p_(h_myopic,-rho_exante) + b_exante,
                  1/rho_exante
                 ),
nu_myopic = h_myopic * kappa_myopic,
unit_cost_myopic = qacac * kappa_myopic + wagerate * nu_myopic,
psi_myopic_hat = unit_cost_myopic/unit_cost_myopic(-1)-1,
psi_hat = psi_non_myopic/psi_non_myopic(-1)-1,

   {because psi_hat is derived from the rate of change of current unit production cost
   on the newest vintage, the factor /(1+r) has to be added in order to arrive at a
   present value price equivalent of psi_hat
}

r1 = p_((1+w_hat)/(1+r),1-1/(1+rho_expost)) *
        p_(((1+psi_hat)/(1+r))*
          p_(1+if_else_(and_(experiment4,
                        and_(ge_(time,start_exp),le_(time,stop_exp))),
                    gamma_n+d_gamma_n,
                    gamma_n),
            -rho_expost),
          1/(1+rho_expost))*(1-delta),

r2 = (1+psi_hat)*(1-delta)/(1+r), {!!!!!}

s1 = if_else_(and_(experiment1,
                  and_(ge_(time,start_exp),le_(time,stop_exp))),
              1,
              1/(1-r1)),
s2 = if_else_(and_(experiment1,
                  and_(ge_(time,start_exp),le_(time,stop_exp))),
              1,
              1/(1-r2)),
z = p_(wagerate/psi_non_myopic,1 - 1/(1+rho_expost)) * s1/s2,
nu_bar =p_(a_exante * p_(z,-1-1/rho_expost), 1/(1+rho_exante)),
kappa_bar = p_((1-a_exante * p_(nu_bar, - rho_exante))/b_exante, -1/rho_exante),
a_expost = a_exante * p_(nu_bar,   rho_expost - rho_exante),
b_expost = b_exante * p_(kappa_bar, rho_expost - rho_exante),
h = p_(b_expost * wagerate/(a_expost*qacac), -1/(1+rho_expost)),

    {nu0 and kappa0 are entry points on new ex post production function}

nu0   = p_(a_expost + b_expost * p_(h, rho_expost),1/rho_expost),
kappa0 = p_(a_expost * p_(h, - rho_expost) + b_expost, 1/rho_expost),
psi_non_myopic = nu0 * wagerate + kappa0 * qacac,
```

```
    {cf equation 38, note that all existing techniques are (by definition) always on an
    ex post iso-quant
}

xi = p_(a_expost, -1/(1+rho_expost)) * nu0,

    {nux is labour coefficient at t of vintage installed at time t - x, i.e. a vintage of x
    years old. Because of disembodied tc => a_expost(i) has changed since the time of
    installation. Note that only the distribution parameters are vintage specific, and
    NOT THE SUBSTITUTION parameters}

dot_(dt=1 to 99 :
    nu.= p_(a_expost(dt)*
         p_(cum_dis(time-dt,
                time,
                gamma_n,
               if_else_(and_(experiment4,
                      and_(ge_(time,start_exp),le_(time,stop_exp))),
                      gamma_n+d_gamma_n,
                      gamma_n),
                start_exp,
                stop_exp),
           -rho_expost),
         1/(1+rho_expost))*xi
    ),

    {kappax see nux above}

dot_(dt=1 to 99:
     kappa. =
  max(0,p_((1-a_expost(dt)*
         p_(cum_dis(time-dt,
                time,
                gamma_n,
                if_else_(and_(experiment4,
                       and_(ge_(time,start_exp),le_(time,stop_exp))),
                       gamma_n+d_gamma_n,
                       gamma_n),
                start_exp,
                stop_exp),
           -rho_expost)*
         p_(nu.,-rho_expost))/
         (b_expost(dt)*p_(1+gamma_i,-rho_expost*dt)),-1/rho_expost))
    ),

    {xz production capacity vintage z at time t}

dot_(dt=1 to 99:x.= if_else_(le_(time-abs(dt),100.1),0,
                      (p_(1-delta,abs(dt))*inv(dt))/kappa.)),
xold = sum_(i=-99 to -1 : ref_(x,i)),
```

```
    {nz capacity labour demand vintage z at time t}

dot_(dt=1 to 99:n.=nu.*x.),
nold = sum_(i=-99 to -1:ref_(n,i)),

    {xold and nold contain production capacity and capacity labour demand, respect-
    ively, of existing vintages}


    {x is total exogenously given production capacity}

y = x,

    {determine size newest vintage}

x0=max(0,y-xold),

    {determine required amount of capital}

inv = x0 * kappa0,
n0 = nu0 * x0,

    {total capacity labour demand}

n_vin = nold + n0,

    { vintage capital stock}

k_vin = sum_(i=-99 to 0 : if_else_(le_(time-abs(i),100.1),
                          0,
                          p_(1-delta,abs(i))*inv(i))),
inv_vin = inv,

    {now updating of average capital productivity and average labour intensity}

inv_rum = inv,
k_rum = if_else_(le_(time,100.1),0,(1 - delta)* k_rum(-1) + inv_rum),
weight=if_else_(le_(time,101.1),0,(1-delta)*k_rum(-1)/k_rum),
t1 = t1(-1)*weight*p_(1/(1+gamma_i),-rho_expost) +
       (1-weight)/b_expost,
t2 = t2(-1)*weight*
       p_(p_(1+if_else_(and_(experiment4,
                  and_(ge_(time,start_exp),le_(time,stop_exp))),
                  gamma_n+d_gamma_n,
                  gamma_n),
           1/(1+rho_expost))/(1+gamma_i),
         -rho_expost) +
       (1-weight) * p_(a_expost,1/(1+rho_expost))/b_expost,
```

```
t3 = t3(-1)*weight*
        p_(p_(1+if_else_(and_(experiment4,
                     and_(ge_(time,start_exp),le_(time,stop_exp))),
                     gamma_n+d_gamma_n,
                     gamma_n),
            rho_expost/(1+rho_expost))/(1+gamma_i),
          -rho_expost)+
        (1-weight)*p_(a_expost, rho_expost/(1+rho_expost))/b_expost,
t4 = t4(-1)*weight*
        p_((1+if_else_(and_(experiment4,
                   and_(ge_(time,start_exp),le_(time,stop_exp))),
                   gamma_n+d_gamma_n,
                   gamma_n))/(1+gamma_i),
           -rho_expost) +
        (1-weight)*a_expost/b_expost,
zeta_rum = p_(t1 - p_(xi, -rho_expost)* t2, 1/rho_expost),
theta_rum = p_(p_(xi, rho_expost) * t3 - t4, 1/rho_expost),

    {calculate RUM aggregates + growthrates}

n_rum = theta_rum * k_rum,
x_rum = zeta_rum * k_rum,
zeta_vin = y/k_vin,
theta_vin  = n_vin/k_vin,
zeta_error = 100*(zeta_rum/zeta_vin-1),
theta_error = 100*(theta_rum/theta_vin-1),
g_zeta_rum = 100*(zeta_rum/zeta_rum(-1)-1),
g_zeta_vin = 100*(zeta_vin/zeta_vin(-1)-1),
g_theta_rum = 100*(theta_rum/theta_rum(-1)-1),
g_theta_vin = 100*(theta_vin/theta_vin(-1)-1),

    { calculate capital productivity growth newest vintage}

g_zeta_0 = (kappa0(-1)/kappa0-1)*100,
{calculate growth labour/capital ratio newest vintage}
g_theta_0 = ((nu0/kappa0)/(nu0(-1)/kappa0(-1))-1)*100,

    {calculate share of newest vintage in total capital stock}

sh_newvin_vin = ((inv_vin/kappa0)/y)*100,
sh_newvin_rum = ((inv_rum/kappa0)/x_rum)*100,
list_(n=n_vin,n_rum,nu_myopic,nu_bar,a_expost,b_expost,nu0,kappa_bar,kappa0),
g_#n# = 100*(#n#/#n#(-1)-1),
```

References

- Eigenraam, C.J.J., 1987, 'Een produktiefunctie voor heterogene jaargangen en daarbij behorende factorvraagvergelijkingen', Onderzoeksmemorandum Centraal Planbureau, No. 34, Den Haag.
- Gelauff, G.M.M., Wennekers, A.R.M. and A.H.M. de Jong, 1985, 'A putty-clay model with three factors of production and partly endogenous technical progress', De Economist, Vol. 133, pp. 327-351.
- Hartog, H. den, and H.S. Tjan, 1976, 'Investment, Wages, Prices and Demand for Labour', de Economist, 124, pp. 32-55.
- Kuipers, S.K. and A.H. van Zon, 1982, 'Output and Employment Growth in the Netherlands in the Postwar Period: A Putty-Clay Approach', de Economist, 130, pp. 38-70.
- Malcomson, J.M., 1975, 'Replacement and the rental value of capital equipment subject to obsolescence', Journal of Economic Theory, 10, pp. 24-41.
- Meijers, H. and A.H. van Zon, 1991, 'A Reformulation of the HERMES Vintage Production Structure', MERIT Research Memorandum, no. 91-016.
- Muysken, J. and A.H. van Zon, 1987, 'Employment and unemployment in the Netherlands, 1960-1984 : a putty-clay approach', Recherches Economiques de Louvain, Vol. 53, pp. 101-133.
- Zon, A.H. van and J. Muysken, 1992A, 'Matching Skill Supply and Demand : A Linear Programming Approach', MERIT Research Memorandum, nr. 92-005.
- Zon, A.H. van, and J. Muysken, 1992B, 'MASTER and Technological Change : Some Comparative Model Simulations', MERIT Research Memorandum, no. 92-012.
- Zon, A.H. van and J. Muysken, 1992C, 'MASTER, a Model for the Analysis of Sectoral Technology and Employment Relations : It's Basic Features', MERIT Research Memorandum, 92-017.
- Zon, A.H. van, 1992D, 'The Quasi-Putty-Putty Production Model: A Note on its Features and Uses', MERIT Research Memorandum, no. 92-016.
- Zon, A.H. van, 1993, 'The Continuous Recursive Adjustment Putty-Putty Model: An Outline of Its Main Features', MERIT Research Memorandum, 93-016.

Table of Contents